MATHEMATICAL METHODS IN ELECTROMAGNETIC THEORY

The Lost "Second Lorentz Theorem" in the Phasor Domain

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ABSTRACT: The fundamental relationship of complex conjugate phasors is analyzed for two independent sources of a time harmonic field in comparison with the well-known reciprocity theorem, the Lorentz lemma and the reaction theorem. It has been shown that this relationship in the case of isotropic lossless medium yields a complementary reciprocity theorem, whereas in the specific case of a bianisotropic medium it determines the "pseudo-reciprocity" property. This universal law seems to be useful in the theory of guided waves, antenna theory, etc.

INTRODUCTION

Two energy laws, which are established by the remarkable Poynting theorem and the Lorentz reciprocity theorem (see, e.g. [1-4]), are of key importance in the theory of electromagnetism. The standard transition in describing a monochromatic field from the time domain to the phasor domain results in doubling the number of power relations. This fact in an evident way follows from the representation of the product of time-harmonic field components $\vec{E}_\alpha(t), \vec{H}_\beta(t)$ via phasors $\vec{E}_\alpha(\omega), \vec{H}_\beta(\omega)$, similar, for example, to the following expression for the cross product

$$\vec{E}_\alpha(t)\times\vec{H}_\beta(t)=\frac{1}{2}\mathrm{Re}\left\{\vec{E}_\alpha(\omega)\times\vec{H}^*_\beta(\omega)\right\}+\frac{1}{2}\mathrm{Re}\left\{\left[\vec{E}_\alpha(\omega)\times\vec{H}_\beta(\omega)\right]e^{\pm i2\omega t}\right\}.$$

Here and below the subscripts α and β label the independent sources of the field and the asterisk stands for complex conjugation.

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When going over to the phasor domain the Poynting theorem generates two basic relationships, namely, the famous theorem of complex power and the theorem of oscillating power [4]. As for the pair of the fundamental laws induced by the Lorentz theorem, the well-known reciprocity theorem includes the primary phasors, whereas the second available relation containing the complex conjugate phasors has not a conventional name. In what follows these two laws will be referred to as the first and the second Lorentz theorem, respectively. Such terminology is based on the fact that the latter theorem yields the second reciprocity relation for isotropic lossless media and establishes a new "pseudo-reciprocity" property for anisotropic materials.

Many books are surprisingly fuzzy on this doubling of the fundamental power relations in the phasor domain. The majority of manuals consider only the theorem of complex power and the (first) Lorentz reciprocity theorem. As a rare exception we can note, for example, the treatise [4] where the theorem of oscillating power has been considered as well. It should be noted that all the pairs of the laws stated above have been derived from the Maxwell's equations in the same manner.

The second Lorentz theorem is not so well-known. Particular cases of this theorem have been used earlier in papers [5-7] where some problems of mode diffraction by waveguide discontinuities were considered. As was shown in papers [7,8], the use of the second Lorentz theorem is a key point in deriving the operator form of the power conservation statement. Indeed, the theorem of complex power regards the field from a single source which allows determining only the principal diagonal of the sought for operator matrix. The second Lorentz lemma relates the fields of two independent sources which makes it possible to form all off-diagonal blocks and thus to completely construct the operator matrix of the generalized power conservation law.

The present paper is aimed at a comparative analysis of the basic relations for the general case of a bianisotropic medium, which form the content of the first and second Lorentz theorems, and also of their corollaries known as the Lorentz lemma and the reaction theorem.

In the paper we do not dwell on details of deriving the second Lorentz theorem from the phasor form of the Maxwell's equations. The reason is that the derivation procedure is quite ordinary and can be easily implemented following the scheme which is available in the above references. Instead of that we present a paired record of the familiar and new formulas in order to show in what they are similar and different. In what follows formulas related to the first (reciprocity) Lorentz theorem and its corollaries are marked by a number with the letter "a", while all the new relations are labeled by "b".

COMPARISON OF THE TWO LORENTZ THEOREMS

Let two electromagnetic fields be produced by two completely independent time-harmonic sources α and β separated in space and generating at the same frequency, $\omega_\alpha = \omega_\beta = \omega$. Consider a passive bianisotropic medium for which the constitutive equations for the field phasors can be represented as

$$\vec{D}_\gamma = \vec{\vec{\varepsilon}}\,\vec{E}_\gamma + \vec{\vec{\xi}}\,\vec{H}_\gamma; \qquad \text{with } \gamma = \alpha, \beta. \qquad \vec{B}_\gamma = \vec{\vec{\mu}}\,\vec{H}_\gamma + \vec{\vec{\zeta}}\,\vec{E}_\gamma;$$

Here the dyadic constitutive parameters $\vec{\vec{\varepsilon}}, \vec{\vec{\mu}}, \vec{\vec{\xi}}$ and $\vec{\vec{\zeta}}$ are specified at the frequency ω and may depend on space coordinates.

Following the usual technique (see, e.g. [1-4]), the Maxwell's equations in the phasor form yield two relations

$$div\left[\vec{E}_\alpha \times \vec{H}_\beta - \vec{E}_\beta \times \vec{H}_\alpha\right] = \left[\vec{J}_\alpha \cdot \vec{E}_\beta - \vec{M}_\alpha \cdot \vec{H}_\beta\right] - \left[\vec{E}_\alpha \cdot \vec{J}_\beta - \vec{H}_\alpha \cdot \vec{M}_\beta\right]; \tag{1a}$$

$$div\left[\vec{E}_\alpha \times \vec{H}^*_\beta + \vec{E}^*_\beta \times \vec{H}_\alpha\right] = -\left[\vec{J}_\alpha \cdot \vec{E}^*_\beta + \vec{M}_\alpha \cdot \vec{H}^*_\beta\right] - \left[\vec{E}_\alpha \cdot \vec{J}^*_\beta + \vec{H}_\alpha \cdot \vec{M}^*_\beta\right]. \tag{1b}$$

Here $\vec{J}_{\alpha(\beta)}$ is the conduction current density and $\vec{M}_{\alpha(\beta)}$ is the magnetic current density. Eq. (1a) is valid for a reciprocity medium whose properties are determined by the constitutive parameters

$$\vec{\vec{\varepsilon}}^T = \vec{\vec{\varepsilon}}, \quad \vec{\vec{\mu}}^T = \vec{\vec{\mu}}, \quad \vec{\vec{\xi}}^T = -\vec{\vec{\zeta}}, \tag{2a}$$

whereas the second relation Eq. (1b) holds under the conditions

$$\vec{\vec{\varepsilon}}^\dagger = \vec{\vec{\varepsilon}}, \quad \vec{\vec{\mu}}^\dagger = \vec{\vec{\mu}}, \quad \vec{\vec{\xi}}^\dagger = \vec{\vec{\zeta}}. \tag{2b}$$

Here the superscript T stands for transposition procedure and the dagger "†" denotes the Hermitian conjugation.

The equality Eq. (2b) determines a property of a nonreciprocal material medium which can be referred to as "pseudo-reciprocity". For example, an gyrotropic lossless medium can be described by the relations $\vec{\vec{\varepsilon}}^\dagger = \vec{\vec{\varepsilon}}$ or $\vec{\vec{\mu}}^\dagger = \vec{\vec{\mu}}$. As follows from Eqs. (2a) and (2b), the relations Eqs. (1a) and (1b) both hold simultaneously in the case of an isotropic lossless medium when ε and μ are real-valued scalars. Note that the right-hand side of the standard equality Eq. (1a) has meaning of the oscillating power density, whereas the right-hand side of the second equality Eq. (1b) involves the complex power density.

At those points where no field sources are present the following two corollaries of the basic relations Eqs. (1a) and (1b) are met,

$$div\left[\vec{E}_\alpha \times \vec{H}_\beta - \vec{E}_\beta \times \vec{H}_\alpha\right] = 0; \tag{3a}$$

$$div\left[\vec{E}_\alpha \times \vec{H}^*_\beta + \vec{E}^*_\beta \times \vec{H}_\alpha\right] = 0\,. \tag{3b}$$

These equalities will be referred to as the first and, respectively, the second Lorentz lemma in the differential form.

The two Lorentz lemmas Eqs. (3a) and (3b) are also valid when magnetic currents are absent and the Ohm law holds, $\vec{J}_\gamma = \vec{\vec{\sigma}}\,\vec{E}_\gamma$, $\gamma = \alpha, \beta$, and the conductivity affinor shows the corresponding property as follows

$$\vec{\vec{\sigma}} = \vec{\vec{\sigma}}^T; \tag{4a}$$

$$\vec{\vec{\sigma}} = -\vec{\vec{\sigma}}^\dagger\,. \tag{4b}$$

To represent the Lorentz theorems in an integral form, let us consider a closed nonpathological surface S enclosing a volume V of the material medium.

Following [3] the results obtained can be represented in terms of the Rumsey reactions. To that end let us introduce the six-component row-vector to describe the fields, $\mathbf{f}_\gamma = \left\{\vec{E}_\gamma, i\,\vec{H}_\gamma\right\}$, $\gamma = \alpha, \beta$, and the row-vector for the given sources, $\mathbf{g}_\gamma = \left\{\vec{J}_\gamma, i\,\vec{M}_\gamma\right\}$, $\gamma = \alpha, \beta$. Treating the reaction concept in the phasor domain we can obtain scalar products of two types, viz.

$$\left(\mathbf{f}_\gamma, \mathbf{g}^T_\gamma\right) \equiv \int_V \mathbf{f}_\gamma \cdot \mathbf{g}^T_\gamma dV = \int_V \left[\vec{E}_\gamma \cdot \vec{J}_\gamma - \vec{H}_\gamma \cdot \vec{M}_\gamma\right] dV; \tag{5a}$$

$$\left(\mathbf{f}_\gamma, \mathbf{g}^\dagger_\gamma\right) \equiv \int_V \mathbf{f}_\gamma \cdot \mathbf{g}^\dagger_\gamma dV = \int_V \left[\vec{E}_\gamma \cdot \vec{J}^*_\gamma + \vec{H}_\gamma \cdot \vec{M}^*_\gamma\right] dV\,, \tag{5b}$$

which will be referred to as the Rumsey reaction and the complex reaction, respectively.

By applying the divergence theorem to the relations Eqs. (1a) and (1b) we can find a general form for both the Lorentz reciprocity theorem and the second Lorentz theorem, respectively, viz.

$$\left(\mathbf{f}_\alpha, \mathbf{g}^T_\beta\right) - \left(\mathbf{g}_\alpha, \mathbf{f}^T_\beta\right) = -\oint_S \left[\vec{E}_\alpha \times \vec{H}_\beta - \vec{E}_\beta \times \vec{H}_\alpha\right] d\vec{S}\,; \tag{6a}$$

$$\left(\mathbf{f}_\alpha, \mathbf{g}_\beta^\dagger\right) + \left(\mathbf{g}_\alpha, \mathbf{f}_\beta^\dagger\right) = -\oint_S \left[\vec{E}_\alpha \times \vec{H}_\beta^* + \vec{E}_\beta^* \times \vec{H}_\alpha\right] d\vec{S}\,, \tag{6b}$$

where $d\vec{S} \equiv \vec{n}\, dS$, with $\vec{n}$ being the outward normal to the surface.

Two specific cases of the basic relation Eq. (6a) seem to be special interest, which are the integral form of the first Lorentz lemma Eq. (3a) and the reaction theorem [3,4].

The first and second reaction theorems can be written as

$$\left(\mathbf{f}_\alpha, \mathbf{g}_\beta^T\right) = \left(\mathbf{f}_\beta, \mathbf{g}_\alpha^T\right); \tag{7a}$$

$$\left(\mathbf{f}_\alpha, \mathbf{g}_\beta^\dagger\right) = -\left(\mathbf{f}_\beta, \mathbf{g}_\alpha^\dagger\right)^\dagger, \tag{7b}$$

respectively. They follow in evident way from the integral equalities Eqs. (6a) and (6b) when the volume V under consideration represents the entire unbounded space containing the sources and inhomogeneities of a finite scale-size. These reaction theorems are valid for bounded volumes V as well, provided that the following equalities hold

$$\oint_S \left[\vec{E}_\alpha \times \vec{H}_\beta - \vec{E}_\beta \times \vec{H}_\alpha\right] d\vec{S} = 0\,; \tag{8a}$$

$$\oint_S \left[\vec{E}_\alpha \times \vec{H}_\beta^* + \vec{E}_\beta^* \times \vec{H}_\alpha\right] d\vec{S} = 0\,. \tag{8b}$$

In turn, two conditions Eqs. (8a) and (8b) will be *a fortiori* met with the impedance boundary conditions for the tangential electric field component,

$$\left.\vec{E}_\tau\right|_S = W\left[\vec{n} \times \vec{H}\right]_S, \tag{9}$$

where the impedance W assumes any arbitrary value in the case of the equality Eq. (8a) (see, e.g. [3]) and provided that $\mathrm{Re}\,W = 0$ in the case of Eq. (8b). Note that the complex-valued impedance in Eq. (9) can be dependent on the position on the surface S.

Finally, the relations Eqs. (8a) and (8b) coincide with the integral forms of the first and second Lorentz lemmata, respectively, provided that the above mentioned conditions are met and when the terms corresponding to the Rumsey reaction in Eq. (5a) and to the complex reaction in Eq. (5b) vanish.

CONCLUSIONS

When solving electrodynamic problems in the phasor domain it should be taken into account two Lorentz theorems are available which are different in the form and sense. These two fundamental laws have been referred above as the first Lorentz (reciprocity) and the second Lorentz (pseudo-reciprocity) theorems. As a result, we have two Lorentz lemmata and two reaction theorems provided that the special conditions mentioned above are met.

As has been shown, the first and the second Lorentz theorems are valid for a variety of media which show, generally speaking, absolutely different properties. In contrast to the Lorentz reciprocity theorem, the second theorem, which is also valid for nonreciprocal media, corresponds to a property which has been defined as the "pseudo-reciprocity". However in the case of isotropic lossless media the theorems are both valid, and hence we have in this situation two complementary reciprocity relations.

Besides its significance for the theory, the second Lorentz theorem seems to be important for the analytical justification of the methods which are based on the modal analysis. Specifically, application of the theorem to solving the problem of mode diffraction makes it possible to formulate the power conservation law in the operator form [6-8]. It is beyond any doubt that this lost universal law will find another important application in the applied electrodynamics.

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