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TIME-DOMAIN ANALYSIS OF OPEN RESONATORS. ANALYTICAL GROUNDS

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Abstract—The paper is concerned with the development and mathematical justification of the methodology for applying the time-domain methods in the study of spectral characteristics of open electrodynamic resonant structures.

1. INTRODUCTION

This paper is concerned with the solution of theoretical and methodological problems arising in the time-domain (TD) analysis of open resonators (OR). The potentialities of the frequency-domain (FD) methods as applied to the problems of this kind have been almost exhausted. This became evident when researches came up against the synthesis problems for open dispersive structures in resonant quasi-optics, for electrodynamic systems in devices of solid-state or vacuum electronics, and others. The application of more universal TD methods [1, 2] to the analysis of resonance conditions, which are acutely sensitive to variations in the system parameters, must be grounded on stable and reliable numerical algorithms. At the same time, the physical treatment of the numerical results here is impossible without justified and simplified analytical representations for the solutions of the initial boundary-value problems considered. In numerical experiments, proper allowance must be made for the results

obtained in FD. We propose to consider the basic stages of the study of OR by the example of the transient E-polarized fields in the near zone of compact inhomogeneities of the $\boldsymbol{R}^2$-space. Similar reasoning holds in the case of H-polarization and in 3-D vector case as well, and for OR of other types [3].

We discuss the following points:

- rigorous formulation and solution of initial boundary-value problems in the theory of OR (the finite difference method with 'fully absorbing' conditions on the artificial boundaries of the analysis domain);
- well-grounded analytical relations between spatial-temporal and spatial-frequency representations for the solutions;
- the problem of choice of the pulsed source in numerical experiments;
- the general methodology for analyzing OR by TD methods.

2. INITIAL BOUNDARY-VALUE PROBLEMS OF THE THEORY OF OPEN COMPACT RESONATORS

We consider a two-dimensional initial boundary-value problem

$$\begin{cases} P[U] \equiv \left[-\varepsilon(g)\dfrac{\partial^2}{\partial t^2} - \sigma(g)\dfrac{\partial}{\partial t} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right] U(g,t) = F(g,t) \equiv \dfrac{\partial J_x}{\partial t}, \\ g = \{y,z\} \in \boldsymbol{Q}, \quad t > 0 \\ U(g,t)|_{g\in\boldsymbol{S}} = 0, \quad t \geq 0 \\ U(g,0) = \varphi(g), \ \left.\dfrac{\partial U(g,t)}{\partial t}\right|_{t=0} = \psi(g), \quad g \in \overline{\boldsymbol{Q}} \end{cases}, \tag{1}$$

describing transient states of E-polarized ($\partial/\partial x \equiv 0$, $U(g,t) = E_x(g,t)$, $E_y = E_z = H_x = j_y = j_z = 0$, $\partial H_y/\partial t = -\eta_0^{-1}\partial E_x/\partial z$, and $\partial H_z/\partial t = -\eta_0^{-1}\partial E_x/\partial y$) electromagnetic waves $\{\vec{E}, \vec{H}\}$ in compact open resonators, whose geometry (Fig. 1) is specified by the real-valued finite functions $\varepsilon(g) - 1$, $\sigma(g)$ and by the contours $\boldsymbol{S}$ that are the boundaries of the domains $\mathrm{int}\boldsymbol{S}$ filled with a perfect conductor ($\boldsymbol{Q} = \boldsymbol{R}^2 \backslash \overline{\mathrm{int}\boldsymbol{S}}$).

Here $g = \{y,z\}$ is a point in $\boldsymbol{R}^2$-space, $\overline{\boldsymbol{G}}$ is the closure of the domain $\boldsymbol{G}$, $\vec{E} \equiv \vec{E}(g,t)$ and $\vec{H} \equiv \vec{H}(g,t)$ are the electric and magnetic field vectors, $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the free space impedance, ε_0 and μ_0 are the electrical and magnetic constants, $\vec{J} = \eta_0\vec{j}$ ($\vec{j} \equiv \vec{j}(g,t)$) is the extraneous current density, $\sigma = \eta_0\sigma_0$, $\varepsilon \equiv \varepsilon(g) \geq 1$

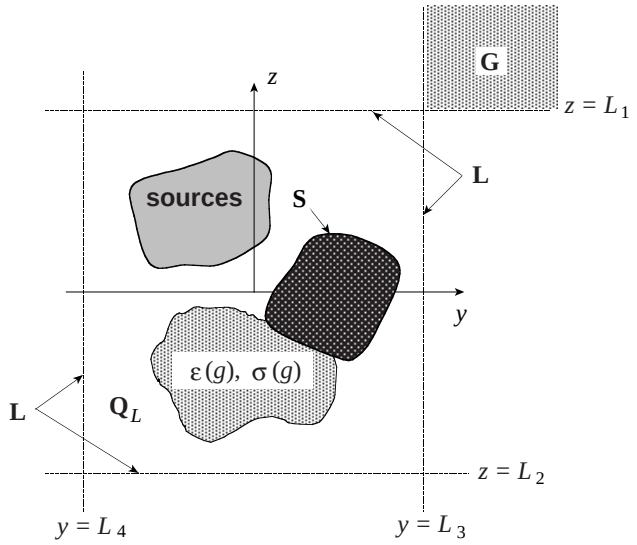

Figure 1. Geometry of the problem (1).

and $\sigma_0 \equiv \sigma_0(g) \geq 0$ are the relative permittivity and the specific conductivity of a locally inhomogeneous, isotropic, nonmagnetic, and nondispersive propagation medium. The functions $F(g,t)$ describing 'current' sources as well as the functions $\varphi(g)$ and $\psi(g)$ describing 'momentary' sources are finite in $\overline{\boldsymbol{Q}}$.

In the paper, the SI system of units is used. The variable t is the product of the real time by the velocity of light in free space and has the dimensions of length.

Statement 1 [4]. *Let* $F(g,t) \in \boldsymbol{L}_{2,1}(\boldsymbol{Q}^T)$, $\varphi(g) \in \dot{\boldsymbol{W}}_2^1(\boldsymbol{Q})$, $\psi(g) \in \boldsymbol{L}_2(\boldsymbol{Q})$, $\boldsymbol{Q}^T = \boldsymbol{Q} \times (0;T)$, $(0;T) = \{t : 0 < t < T < \infty\}$, *while the functions* $\partial\varepsilon(g)/\partial y$, $\partial\varepsilon(g)/\partial z$ *and* $\sigma(g)$, $g \in \boldsymbol{Q}$ *be bounded. Then problem (1) for all* $t \in [0;T]$ *has a generalized solution from the energy class, and the uniqueness theorem is true in this class.* $\triangleleft$

By a generalized solution from the energy class we understand a function $U(g,t)$, belonging to $\dot{\boldsymbol{W}}_2^1(\boldsymbol{Q})$ for any $t \in [0;T]$ and depending continuously on t in the norm $\boldsymbol{W}_2^1(\boldsymbol{Q})$. Furthermore the derivative $\partial U/\partial t$ should exist as an element of the space $\boldsymbol{L}_2(\boldsymbol{Q})$ for any $t \in [0;T]$ and vary continuously with t in the norm $\boldsymbol{L}_2(\boldsymbol{Q})$. The initial conditions in (1) should be continuous in the spaces $\dot{\boldsymbol{W}}_2^1(\boldsymbol{Q})$ and $\boldsymbol{L}_2(\boldsymbol{Q})$, respectively, while the equation of telegraphy be satisfied

in terms of the identity

$$\int\limits_{\boldsymbol{Q}^T} \left\{ \varepsilon \left(\frac{\partial}{\partial t} U\right) \left(\frac{\partial}{\partial t} \gamma\right) - \sigma \left(\frac{\partial}{\partial t} U\right) \gamma - \left(\frac{\partial}{\partial y} U\right) \left(\frac{\partial}{\partial y} \gamma\right) - \left(\frac{\partial}{\partial z} U\right) \left(\frac{\partial}{\partial z} \gamma\right) \right\} dgdt + \int\limits_{\boldsymbol{Q}} \varepsilon \psi \gamma(g,0) dg = \int\limits_{\boldsymbol{Q}^T} F\gamma dgdt.$$

Here $\gamma = \gamma(g,t)$ is an arbitrary element from $\boldsymbol{W}_{2,0}^{1}(\boldsymbol{Q}^T)$ such that $\gamma(g,T) = 0$. This identity is derived in a formal way from the following identity

$$(P[U] - F, \gamma) = \int\limits_{\boldsymbol{Q}^T} (P[U] - F)\gamma dgdt = 0$$

by means of single partial integration of the terms, containing second order derivatives of the function $U(g,t)$. In [4] it was proven that such a definition makes sense and is actually a generalized notion of the classic solution.

Under practically the same assumptions, the unique solvability of the problem (1) and problems (1) with the impedance type boundary conditions in the space $\boldsymbol{W}_{2}^{1}(\boldsymbol{Q}^T)$ has been proved in [4]. The class of generalized solutions, that has been called the energy class, is somewhat narrower than the class of generalized solutions from $\boldsymbol{W}_{2}^{1}(\boldsymbol{Q}^T)$. However, it can be proved for this class that the solution $U(g,t)$ has the same differential features that are assumed satisfied at the initial time (continuable initial conditions). Besides, for a $U(g,t)$ from this class the following energy relation is satisfied

$$\int\limits_{\boldsymbol{Q}} \left(\varepsilon \left(\frac{\partial U}{\partial t}\right)^2 + |\operatorname{grad} U|^2 \right) dg \Bigg|_0^T + 2 \int\limits_{\boldsymbol{Q}^T} \left(\sigma \left(\frac{\partial U}{\partial t}\right)^2 + \left(F \frac{\partial U}{\partial t} \right) \right) dg\, dt = 0. \tag{2}$$

3. FINITE-DIFFERENCE METHOD

The finite difference method reduces problem (1) with $\sigma \equiv 0$ and $t \in [0;T]$ to determining the mesh functions $u = U(y_j, z_k, t_m) = U(j,k,m)$ that satisfy the difference equations

$$\left[-\varepsilon(j,k) D_+^t D_-^t + D_+^y D_-^y + D_+^z D_-^z \right] u = F(j,k,m) \tag{3}$$

at the mesh points $g_{jk} = \{y_j, z_k\} \in \boldsymbol{Q}(h,T)$ on the time layers $t_m = ml,\ m = 0,1,\ldots,M-1 = T/l$. They are complemented by

the equations

$$\begin{cases} U(j,k,0)=\varphi(j,k),\ U(j,k,1)=\varphi(j,k)+l\psi(j,k),\ g_{jk}\in \boldsymbol{Q}(h,T) \\ U(j,k,m)=0,\ g_{jk}\in S(h,T),\ m=0,1,\ldots,M-1 \end{cases} \tag{4}$$

(i.e., the difference analogues of the initial and boundary conditions in (1)). Here $D^y_+[u]=h^{-1}[U(j+1,k,m)-U(j,k,m)]$ and $D^y_-[u]=h^{-1}[U(j,k,m)-U(j-1,k,m)]$ are the standard operators of the right- and left-hand difference derivatives (the same with obvious changes is true also for $D^z_\pm[u],\ D^t_\pm[u]$); $y_j=jh,\ z_k=kh,\ j,k=0,\pm1,\ldots;\ h>0$ and $l>0$ are the space-step and the time-step of the mesh; all mesh functions $f(j,k)$ at the mesh points $g_{jk}\in \boldsymbol{Q}(h,T)$ are constructed with respect to $f(g),\ g\in \boldsymbol{Q}$ as the averages

$$f(j,k)=h^{-2}\int\limits_{\omega_h(j,k)} f(g)dg,$$

$$\omega_h(j,k)=\{g: jh<y<(j+1)h; kh<z<(k+1)h\},$$

$\boldsymbol{Q}(h,T)$ is the union of cells $\omega_h(j,k)$ belonging to $\boldsymbol{Q}(T)$; $\boldsymbol{S}(h,T)$ is the boundary of $\boldsymbol{Q}(h,T)$; $\boldsymbol{Q}(T)$ is the cut of the cone of influence of sources $F(g,t),\ \varphi(g)$, and $\psi(g)$ in the region $\boldsymbol{Q}$ at the time $t=\tau>T$. It is obvious that equations (3) and (4) uniquely determine u, and u can be calculated without inversion of any matrix operators (i.e., through an explicit scheme).

The finite-difference scheme is considered to be stable, if for the approximate solutions u a boundedness can be determined that is uniform with respect to h and l. From the stability, the intrinsic convergence of the sequence $\{u\}_{h,l}$ follows, and the limiting function u will be the solution to the original initial boundary-value problem provided that this problem is approximated by finite difference equations. The latter is satisfied for (1) and for (3), (4). As for the stability of the considered scheme, it is most convenient to analyze it in the 'energy' spaces where the original problem is well posed. In [4], the validity of the following statement has been proven on the basis of difference analogues of the energy inequalities.

Statement 2. *Let the functions $F(g,t),\ \varphi(g),\ \psi(g)$, and $\varepsilon(g)-1$ that are finite in the region $\boldsymbol{Q}$ be such that $F(g,t)\in \boldsymbol{L}_{2,1}(\boldsymbol{Q}^T),\ \varphi(g)\in \boldsymbol{W}^1_2(\boldsymbol{Q}),\ \psi(g)\in \boldsymbol{L}_2(\boldsymbol{Q})$ and $\xi\le\varepsilon^{-1}(g)\le\eta;\ g\in\boldsymbol{Q}$, while the derivatives $\partial\varepsilon(g)/\partial y$ and $\partial\varepsilon(g)/\partial z$ are bounded. Then the norms $\boldsymbol{W}^1_2(\boldsymbol{Q}^T)$ of the continuous multilinear complements $\tilde{u}$ of solutions u to problems (3), (4) (the interpolations of the mesh functions u that are linear in each variable) are uniformly bounded for any h and l that*

satisfy one of the following conditions:

$$\frac{\eta\sqrt{2}}{\sqrt{\xi}}\frac{l}{h} < 1 \quad \text{or} \quad 2\sqrt{\eta}\frac{l}{h} < 1. \tag{5}$$

The sequence $\{\tilde{u}\}_{h,l}$ *converges weakly in* $\boldsymbol{W}_2^1(\boldsymbol{Q}^T)$ *and strongly in* $\boldsymbol{L}_2(\boldsymbol{Q}^T)$ *to the solution* $U(g,t)$ *of the problem (1) as* $h, l \to 0$. $\triangleleft$

4. TRUNCATION OF THE COMPUTATIONAL DOMAIN: "FULLY ABSORBING" CONDITIONS

In Section 3, when describing a general procedure of the algorithmization of problem (1) by the finite-difference method, we have truncated the computational domain by applying the well-known exact radiation condition

$$U(g,t)\Big|_{g\in\boldsymbol{L},\, t\in[0;T]} = 0 \tag{6}$$

for the waves $U(g,t)$ outgoing from the region where the sources and scatterers are localized. The artificial boundary $\boldsymbol{L}$ must be situated, in this case, outside the region $\boldsymbol{G} \subset \boldsymbol{Q}$, whose points are reached by the excitation $U(g,t)$ by the time $t = T$. The principal shortcoming of this approach consists in the necessity to extend the computational domain with increasing T.

The Absorbing Boundary Conditions (see for example [5–8]) and the perfectly-matched absorbing layers [9, 10] allow one to 'close' open initial boundary-value problems by the nearby fixed boundaries $\boldsymbol{L}$, but they distort the simulated processes to some extent. These distortions grow as the observation time t increases [3].

The resonant modes of the nonsinusoidal-wave scattering are highly sensitive to the influence of the virtual fields caused by reflections from the imperfect 'absorbing' boundaries $\boldsymbol{L}$. The calculation of resonance electrodynamic characteristics calls, as a rule, for a long time intervals $0 < t < T$ [11]. Therefore, the reliable analysis of field oscillations in open high-Q resonant structures must not exploit those algorithms that truncate the computational domain inefficiently. In the present paper, the problem is solved with the help of the 'fully absorbing' conditions

$$\left[\frac{\partial}{\partial t} \pm \frac{\partial}{\partial z}\right] U(g,t) = \frac{2}{\pi}\int\limits_0^{\pi/2} \frac{\partial V_1(g,t,\varphi)}{\partial t}\sin^2\varphi d\varphi,\ L_4 \le y \le L_3,\ t \ge 0,$$

$$\begin{cases} \left[\dfrac{\partial^2 V_1(g,t,\varphi)}{\partial t^2} - \dfrac{\partial^2 W_1(g,t,\varphi)}{\partial y^2}\right] = 0, \quad L_4 < y < L_3,\ t > 0 \\ \left.\dfrac{\partial V_1(g,t,\varphi)}{\partial t}\right|_{t=0} = \left. V_1(g,t,\varphi)\right|_{t=0} = 0, \quad L_4 < y < L_3 \end{cases} ; \begin{cases} z = L_1 \\ z = L_2 \end{cases}, \tag{7}$$

$$\left[\frac{\partial}{\partial t} \pm \frac{\partial}{\partial y}\right] U(g,t) = \frac{2}{\pi} \int\limits_0^{\pi/2} \frac{\partial V_2(g,t,\varphi)}{\partial t} \sin^2\varphi d\varphi,\ L_2 \le y \le L_1,\ t \ge 0,$$

$$\begin{cases} \left[\dfrac{\partial^2 V_2(g,t,\varphi)}{\partial t^2} - \dfrac{\partial^2 W_2(g,t,\varphi)}{\partial z^2}\right] = 0, \quad L_2 < y < L_1,\ t > 0 \\ \left.\dfrac{\partial V_2(g,t,\varphi)}{\partial t}\right|_{t=0} = \left. V_2(g,t,\varphi)\right|_{t=0} = 0, \quad L_2 < y < L_1 \end{cases} ; \begin{cases} z = L_3 \\ z = L_4 \end{cases}, \tag{8}$$

$$\begin{cases} \left[\dfrac{\partial}{\partial t} \pm \cos\varphi \dfrac{\partial}{\partial y}\right] W_1(g,t,\varphi) \\ \qquad = \dfrac{2\cos\varphi}{\pi} \displaystyle\int\limits_0^{\pi/2} \frac{\sin^2\gamma}{\cos^2\varphi + \sin^2\varphi\cos^2\gamma} \frac{\partial W_2(g,t,\gamma)}{\partial t} d\gamma \\ \left[\dfrac{\partial}{\partial t} \pm \cos\varphi \dfrac{\partial}{\partial z}\right] W_2(g,t,\varphi) \\ \qquad = \dfrac{2\cos\varphi}{\pi} \displaystyle\int\limits_0^{\pi/2} \frac{\sin^2\gamma}{\cos^2\varphi + \sin^2\varphi\cos^2\gamma} \frac{\partial W_1(g,t,\gamma)}{\partial t} d\gamma \end{cases}, \quad t \ge 0, \tag{9}$$

$$\begin{Bmatrix} + \\ + \end{Bmatrix} \to g = \{L_3, L_1\},\ \begin{Bmatrix} + \\ - \end{Bmatrix} \to \{L_3, L_2\},$$
$$\begin{Bmatrix} - \\ + \end{Bmatrix} = \{L_4, L_1\},\ \begin{Bmatrix} - \\ - \end{Bmatrix} \to \{L_4, L_2\}.$$

Formulas (7)–(9) represent the exact local 'absorbing' condition for the entire artificial coordinate boundary $\boldsymbol{L}$. In this case, $\boldsymbol{L}$ is the boundary of the rectangular domain $\boldsymbol{Q}_L = \{g \in \boldsymbol{Q} : L_4 < y < L_3;\ L_2 < z < L_1\}$ (Fig. 1) enveloping all sources and compact inhomogeneities of $\boldsymbol{R}^2$-space. Conditions (7)–(9), reducing the analysis domain $\boldsymbol{Q}$ of problem (1) down to $\boldsymbol{Q}_L$, have been constructed in [3, 12] without any heuristic assumptions about a fine structure of the field in the vicinity of the artificial boundary. They most closely correspond to the nature of the simulated physical processes. The errors introduced by these

conditions are less by an order than the finite-difference approximation error [3]. Equations (9) play a part of boundary conditions in the auxiliary initial boundary-value problems within formulas (7) and (8). Here $W_j(g,t,\varphi) = V_j(g,t,\varphi)\cos^2\varphi + U(g,t)$ $(j = 1,2)$ and $\left\{\begin{matrix}+\\+\end{matrix}\right\} \rightarrow g = \{L_3, L_1\}$ specifies the signs in the upper and lower equations for different corner points $g = \{y, z\}$.

In essence, formulas (7)–(9) are the analogue of the exact condition (6) in the domain, where the spatial-temporal transformations of an electromagnetic field can be arbitrary in intensity. The boundary $\boldsymbol{L}$ divides the infinite domain $\boldsymbol{Q}$ of the original problem into two ones, namely, $\boldsymbol{Q}_L$ and ${}_L\boldsymbol{Q}$ ($\boldsymbol{Q} = \boldsymbol{Q}_L \bigcup_L \boldsymbol{Q} \bigcup \boldsymbol{L}$). In the former (bounded) region, the standard finite-difference algorithms are used for solving problem (1) with conditions (7)–(9). In the latter region, the field $U(g,t)$ is determined by its values on the boundary $\boldsymbol{L}$ [3, 12].

Statement 3. *Problem (1) in the domain $\boldsymbol{Q}$ and problem (1) in the domain $\boldsymbol{Q}_L = \{g \in \boldsymbol{Q} : L_4 < y < L_3;\ L_2 < z < L_1\}$ with conditions (7)–(9) on its exterior rectangular boundary $\boldsymbol{L}$ are equivalent.*

The internal initial boundary-value problems in (7), (8) are well posed with respect to the auxiliary functions $W_1(g,t,\varphi)$ and $W_2(g,t,\varphi)$. ◁

The Statement validity results from the following three facts. The original problem is uniquely solvable [4]. The solution to the original problem is, at the same time, the solution to the modified problem (in construction). The solution to the modified problem is unique. The last-named fact can be proved by using 'energy' estimates for the real function $U(g,t)$ (see, for example, [4, 13]).

5. SPACE-TIME REPRESENTATIONS FOR TRANSIENT FIELDS

Based on the 'energy' estimates, Statement 1 can be reformulated in terms of the space $\boldsymbol{W}_2^1(\boldsymbol{Q}^\infty, \beta) \equiv \{\{U(g,t)\} : U(g,t)\exp(-\beta t) \in \boldsymbol{W}_2^1(\boldsymbol{Q}^\infty);\ \beta \geq 0\}$. Hence [4, 14], the direct and inverse Laplace transforms

$$\tilde{f}(s) = L[f](s) \equiv \int\limits_0^\infty f(t)e^{-st}dt \leftrightarrow f(t) = L^{-1}[\tilde{f}](t) \equiv \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \tilde{f}(s)e^{st}ds \tag{10}$$

can be applied to relate the solutions of the initial boundary-value problem (1) and the solutions of the elliptic boundary-value problem

$$\begin{cases} \tilde{P}[\tilde{U}] \equiv \left[\dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} + \tilde{\varepsilon}k^2\right]\tilde{U}(g,k,\tilde{f}) = \tilde{f}(g,k), \quad g \in \boldsymbol{Q} \\ \tilde{U}(g,k,\tilde{f})\Big|_{g\in\boldsymbol{S}} = 0 \end{cases} . \quad (11)$$

Here $\tilde{\varepsilon}(g) = \varepsilon(g) + i\sigma(g)/k$, $\tilde{f}(g,k) = \tilde{F}(g,k) + ik\tilde{\varepsilon}(g)\varphi(g) - \varepsilon(g)\varphi(g)$, $\tilde{F}(g,k) \leftrightarrow F(g,t)$, and $s = -ik$.

As is well-known [14–16], for $\operatorname{Im} k > 0$ and for any $\tilde{f}(g,k)$ from $\boldsymbol{L}_2(\boldsymbol{Q})$, problem (11) is uniquely solvable in $\boldsymbol{W}_2^1(\boldsymbol{Q})$, while its resolvent is an analytic operator-function of the parameter k. Suppose $\operatorname{Re} s > \beta \geq 0$ ($\operatorname{Im} k > \beta$) and the function $\tilde{U}(g,k,\tilde{f})$ is absolutely integrable with respect to $\operatorname{Re} k$ on $\boldsymbol{R}^1$ with some $\operatorname{Im} k = \alpha > \beta$, then the solution $U(g,t)$ to problem (1) from the energy class and the solution $\tilde{U}(g,k,\tilde{f})$ to problem (11) from $\boldsymbol{W}_2^1(\boldsymbol{Q})$ are related by the following formulas:

$$U(g,t) = \frac{1}{2\pi}\int\limits_{i\alpha-\infty}^{i\alpha+\infty} \tilde{U}(g,k,\tilde{f})e^{-ikt}dk, \quad \tilde{U}(g,k,\tilde{f}) = \int\limits_0^\infty U(g,t)e^{ikt}dt. \quad (12)$$

A central problem of the approaches based on the space-time representations is to obtain a reliable information about analytical properties of the resolvent operator-function of problem (11) everywhere over the natural region of variation of the complex frequency parameter k. This problem has been discussed in the context of the spectral theory of open resonators. Below are given some results of this theory [14, 15, 17, 18] that will be used in further analysis.

Let us add to problem (11) the radiation condition

$$\tilde{U}(g,k,\tilde{f}) = \sum_{n=-\infty}^{\infty} a_n H_n^{(1)}(k\rho)e^{in\phi} \quad (13)$$

($H_n^{(1)}$ is the Hankel function, $\{\rho,\phi\}$ are the polar coordinates in the $y0z$-plane), which is true for $\operatorname{Im} k > 0$ in the domain ${}_a\boldsymbol{Q} : {}_a\overline{\boldsymbol{Q}} = \boldsymbol{Q}\backslash\boldsymbol{Q}_a$, $\boldsymbol{Q}_a = \{g \in \boldsymbol{Q} : |g| < a\}$ being free from scatterers and sources. This condition generalizes the Sommerfeld condition when extending the elliptic problem into the region of complex k. The natural boundaries of this extension are determined by the infinite sheeted Riemann surface $\boldsymbol{K}$ of the analytical continuation of the

fundamental solution to the Helmholtz equation or, what is the same, of the function $\operatorname{Ln} k$.

Statement 4 [15, 19, 20]. *The resolvent $A^{-1}(k)$ of the problem (11), (13) $A(k)[\tilde{U}(g,k,\tilde{f})] = \tilde{f}(g,k)$ is a meromorphic (in local on the surface $\boldsymbol{K}$ coordinates) operator-function of the complex parameter k. Its leading part $\Xi[A^{-1}(k)]$ can be expanded in the vicinity of the pole $k = \overline{k}$ (in the vicinity of the eigenvalue $k = \overline{k}$ of the operator-function $A(k)$) as follows*

$$\Xi[A^{-1}(k)] = \sum_{j=1}^{J} \sum_{m=1}^{M(j)} (k - \overline{k})^{-m} \sum_{l=0}^{M(j)-m} w_l^{(j)}(\cdot) u_{M(j)-m-l}^{(j)}. \qquad (14)$$

Here

$$u_0^{(j)}(g), u_1^{(j)}(g), \ldots, u_{M(j)-1}^{(j)}(g), \quad j = 1, 2, \ldots, J$$

is a canonical set of the eigen and adjoined elements of the operator-function $A(k)$ associated with the eigenvalue $\overline{k}$. This set determines uniquely the canonical set

$$w_0^{(j)}(g), w_1^{(j)}(g), \ldots, w_{M(j)-1}^{(j)}(g), \quad j = 1, 2, \ldots, J$$

of the eigen and adjoined elements of the operator-function $\overline{A}(k) = [A(k^*)]^*$ (* signifies the conjugation procedure) associated with the eigenvalue $\overline{k}^*$. ◁

Numerical algorithms for solving spectral problems for open compact resonators are grounded on the equivalent reformulation of homogeneous boundary-value problems like (11), (13) to homogeneous operator equations

$$A(k)[u(g,k)] = 0, \quad k \in \boldsymbol{K}, \quad g \in \boldsymbol{Q}_a \qquad (15)$$

with infinite finite-meromorphic matrix-functions $B(k) = A(k) - E : l_2 \to l_2$ (E is a unitary matrix), generating a kernel operator or the Koch matrix. In this case, the determinant $\det[A(k)]$ exists, and the components $\overline{k}$ of the spectrum Ω_k (eigenvalues or eigen frequencies) can be determined with a given accuracy by reducing exact characteristic equations

$$d(k) = \det[A(k)] = 0, \quad k \in \boldsymbol{K}. \qquad (16)$$

The order of root $\overline{k}$ of scalar equation (16) determines the order of the eigenvalue $\overline{k}$ of operator equation (15), in other words, the value $M = M(1) + M(2) + \cdots + M(J)$, where J is the number of linear-independent eigenfunctions $u_0^{(j)}(g)$ (the number of distinctive

free oscillations at the eigen frequency $\overline{k}$), while $M(j)-1$ is the number of the adjoined functions $u_m^{(j)}(g)$ of the eigenfunction of number j. The order of pole of the resolvent $A^{-1}(k)$ (and of the Green function $\tilde{G}(g, g_0, k)$ of problem (11), (13)) is determined for $k = \overline{k}$ by the maximal value of $M(j)$.

In the course of the numerical experiments, problem (16) has been divided according symmetry classes of free oscillations. We have not detected the roots whose order is greater than 1 [17]. If the poles of the resolvent $A^{-1}(k)$ are simple, the increase in the order means the degeneracy of the eigen frequency $\overline{k}$ (one eigenvalue corresponds to a few linear-independent free oscillations in an open resonator). As a rule, this situation does not occur in the physical domain of variables, however, two eigen frequencies (for example, $\overline{k}_1$ and $\overline{k}_2$) associated with free oscillations of different types and of common symmetry class can be close in the metric of the corresponding complex space. The free oscillations begin to 'interact'. As a result, their spectral characteristics undergo considerable local (in the interaction area only) or global changes (see, for example, [17, 21]).

All poles of the resolvent $A^{-1}(k)$ of problem (11), (13) are located below the axis $\operatorname{Im} k = 0$ on the first sheet $\boldsymbol{C}_k$ of the surface $\boldsymbol{K}$ (on the plane $\boldsymbol{C}$ of the complex variable k with a cut along the negative part of the axis $\operatorname{Re} k = 0$). In a number of cases [14, 18], such as a resonator bounded by a sufficiently smooth convex contour $\boldsymbol{S}$ with everywhere positive and finite radius of curvature, the poles depart from the real axis at least logarithmically with increasing $|\operatorname{Re} \overline{k}|$. By deforming the contour of integration in formula (12) downward, we obtain:

$$\begin{aligned} U(g,t) &= \frac{1}{2\pi} \int\limits_{i\alpha-\infty}^{i\alpha+\infty} \left[A^{-1}(k)[\tilde{f}(g,k)]\right] e^{-ikt} dk \\ &= \frac{1}{2\pi} \int\limits_{i\alpha-\infty}^{i\alpha+\infty} \left[\int\limits_{\boldsymbol{Q}} \tilde{G}(g, g_0, k) \tilde{f}(g_0, k) dg_0\right] e^{-ikt} dk \\ &= \frac{1}{i} \left\{ \sum_n \int\limits_{\boldsymbol{Q}} \operatorname*{Res}_{k=\overline{k}_n} \left[\tilde{G}(g, g_0, k) \tilde{f}(g_0, k) e^{-ikt}\right] dg_0 \right. \\ &\quad \left. + \sum_m \int\limits_{\boldsymbol{Q}} \operatorname*{Res}_{k=\underline{k}_m : \underline{k}_m \neq 0} \left[\tilde{G}(g, g_0, k) \tilde{f}(g_0, k) e^{-ikt}\right] dg_0 \right\} + R(g,t) \\ &\qquad\qquad g \in \boldsymbol{Q}_b,\ t > 0. \quad (17) \end{aligned}$$

Here $\boldsymbol{Q}_b$ is some bounded subdomain of $\boldsymbol{Q}$, $\tilde{G}(g, g_0, k)$ is the Green function of problem (11), (13) (the kernel of the operator-function $A^{-1}(k)$), $\overline{k}_n \in \Omega_k$ are the eigenvalues of the operator $A(k)$ (the eigen frequencies of an open compact resonator) located on the first sheet of the surface $\boldsymbol{K}$ above some fixed line $\operatorname{Im} k = c < 0$ and being indexed such that $\operatorname{Im} \overline{k}_{n+1} \le \operatorname{Im} \overline{k}_n$ (their number is limited), $\underline{k}_m$ are the poles of the function $\tilde{f}(g, k)$ differing from the elements of the spectral set Ω_k (it is assumed that they are located in $\boldsymbol{C}_k$ above the line $\operatorname{Im} k = c$).

Taking into account the evident equality $\tilde{\varepsilon}(-k^*) = \tilde{\varepsilon}^*(k)$, the following Statement can be proved [17].

Statement 5.

$$\tilde{G}(g, g_0, k) = \tilde{G}(g_0, g, k) = \tilde{G}^*(g, g_0, -k^*). \quad \triangleleft \tag{18}$$

With account of (18), rewrite (17) in the form

$$U(g,t) = 2\operatorname{Im}\left\{ \sum_n \int_{\boldsymbol{Q}} \operatorname*{Res}_{k=\overline{k}_n : \operatorname{Re}\overline{k}_n > 0} \left[\tilde{G}(g, g_0, k) \tilde{f}(g_0, k) e^{-ikt} \right] dg_0 \right.$$

$$\left. + \sum_m \int_{\boldsymbol{Q}} \operatorname*{Res}_{k=\underline{k}_m : \operatorname{Re}\underline{k}_m > 0} \left[\tilde{G}(g, g_0, k) \tilde{f}(g_0, k) e^{-ikt} \right] dg_0 \right\} + R(g,t)$$

$$g \in \boldsymbol{Q}_b, \ t > 0. \tag{19}$$

We have taken into consideration that $\tilde{f}(g, -k^*) = \tilde{f}^*(g, k)$. This relationship holds for practically all 'current' and 'momentary' sources $F(g,t)$ and $\varphi(g)$, $\psi(g)$.

The term $R(g,t)$ in (19) sums up the contributions of the singularities of $\tilde{U}(g, k, \tilde{f})$, $k \in \boldsymbol{K}$, that are not swept when deforming the contour of integration in (12). The estimate of $R(g,t)$ in the norm of the $\boldsymbol{W}_2^1(\boldsymbol{Q}_b)$-space is determined exclusively by the behavior of $\tilde{U}(g, k, \tilde{f})$ as $k \to 0$ [14]. Thus, for example, if $\tilde{f}(g,k) = O(k^p \ln^q k)$ (p and q are whole), then $\|R(g,t)\| \le \operatorname{const}(\tilde{f})[t^{-p-1} \ln^{q-2} t]$.

In the case, where $\overline{k}$ do not depart from the real axis with increasing $|\operatorname{Re}\overline{k}|$, formula (19) remains true. However, its proof calls for the more detailed analysis of the behavior of $\tilde{U}(g, k, \tilde{f})$ for large $|k|$ [14, 17, 18].

6. A METHODOLOGY FOR ANALYSIS OF OPEN RESONATORS

6.1. Preliminary Qualitative Analysis Based on Rigorous Theoretical Results

Let us formulate several corollaries from (14), (18), (19). Suppose that all poles $k = \overline{k}$ of the Green function $\tilde{G}(g, g_0, k)$ of the problem (11), (13) are simple. Otherwise (if, for example, the point $k = \overline{k},\ \mathrm{Re}\,\overline{k} > 0$ is the second-order pole) the following term

$$2\,\mathrm{Im} \int\limits_{\boldsymbol{Q}} \operatorname*{Res}_{k=k} \left[\tilde{G}(g, g_0, k)\tilde{f}(g_0, k)e^{-ikt}\right] dg_0$$

$$= 2\,\mathrm{Im} \left\{ -ite^{-i\overline{k}t} \int\limits_{\boldsymbol{Q}} G_{-2}(g, g_0, \overline{k}) f_0(g_0, \overline{k}) dg_0 \right.$$

$$\left. + e^{-i\overline{k}t} \int\limits_{\boldsymbol{Q}} \left[G_{-2}(g, g_0, \overline{k}) f_1(g_0, \overline{k}) + G_{-1}(g, g_0, \overline{k}) f_0(g_0, \overline{k})\right] dg_0 \right\}$$

in (19) with $\tilde{f}(g, k) = ik\tilde{\varepsilon}(g)\varphi(g) - \varepsilon(g)\psi(g)$ (an OR is excited by an incoming pulsed wave) will grow faster for $t < T$ than the conservation law (2) admits. From here on, $G_l(g, g_0, \eta)$ and $f_l(g_0, \eta)$ are the coefficients of $(k - \eta)^l$ in the Laurent expansion about $k = \eta$ of the functions $\tilde{G}(g, g_0, k)$ and $\tilde{f}(g_0, k)$.

Without loss of generality it can be assumed that each eigenvalue $\overline{k}$ corresponds to one eigen element $u_0^{(1)}(g) = u(g, \overline{k})$ of the operator-function $A(k)$. The eigen element $w_0^{(1)}(g)$ of the operator-function $\overline{A}(k)$ (Statement 4) that corresponds to the eigenvalue $\overline{k}^*$ is denoted by $w(g, \overline{k}^*)$.

Under the above assumptions, the leading part $\Xi\tilde{G}$ of the Green function $\tilde{G}(g, g_0, k)$ (see (14)) in the vicinity of the eigenvalue $k = \overline{k}$ takes the form

$$\Xi\tilde{G}(g, g_0, k) = \frac{G_{-1}(g, g_0, \overline{k})}{k - \overline{k}} = \frac{u(g, \overline{k})w^*(g_0, \overline{k}^*)}{k - \overline{k}}, \tag{20}$$

while for the eigen elements u and w we have from (18) and (20):

$$u(g, \overline{k}) = w^*(g, \overline{k}^*),\ u(g, \overline{k})w^*(g_0, \overline{k}^*) = -u^*(g, -\overline{k}^*)w(g_0, -\overline{k}). \tag{21}$$

Using (19)–(21), let us give the analytic representation of the solution to problem (1) for some special cases:

A. The function $\tilde{f}(g,k)$ is analytical in $\boldsymbol{C}_k$. From (19) it follows:

$$\begin{aligned} U(g,t) \;&\approx\; 2\,\mathrm{Im}\left[\sum_{n:\mathrm{Re}\,\overline{k}_n>0} u(g,\overline{k}_n)e^{-i\overline{k}_n t}\int\limits_{\boldsymbol{Q}} u(g_0,\overline{k}_n)\tilde{f}(g_0,\overline{k}_n)dg_0\right] \\ &=\; 2\sum_{n:\mathrm{Re}\,\overline{k}_n>0} e^{t\,\mathrm{Im}\,\overline{k}_n}\left|u(g,\overline{k}_n)\right|\left|C(\tilde{f},\overline{k}_n)\right| \\ &\quad \sin\left[\arg u(g,\overline{k}_n)+\arg C(\tilde{f},\overline{k}_n)-t\,\mathrm{Re}\,\overline{k}_n\right]. \end{aligned} \tag{22}$$

From here on $g\in\boldsymbol{Q}_b,\ 0<T_1<t<T$ (T is sufficiently great, while T_1 is dictated by the experimental conditions) and

$$C(f,\overline{k})=\int\limits_{\boldsymbol{Q}} u(g_0,\overline{k})f(g_0,\overline{k})dg_0. \tag{23}$$

In the near-field zone of a compact OR, the field $U(g,t)$ represents a superposition of free oscillations (FO) with complex-valued eigen frequencies $\overline{k}_n$. The rate of damping for each FO is determined by the value of $|\,\mathrm{Im}\,\overline{k}|$ (by the Q-factor $Q=\mathrm{Re}\,\overline{k}/2|\,\mathrm{Im}\,\overline{k}|$ of the oscillation $u(g,\overline{k})$). The initial state (the excitation level) is determined by the value $C(\tilde{f},\overline{k})$, which accounts for a degree of matching of amplitude-spatial and amplitude-frequency characteristics of the functions $u(g,\overline{k})$ and $\tilde{f}(g,k)$.

B. The function $\tilde{f}(g,k)$ possesses one simple pole at the point $k=\underline{k}$ of the right half-plane of the sheet $\boldsymbol{C}_k$ that is not coincide with the members $\overline{k}$ of the set Ω_k. (Wherever a quantity of singularities is concerned, the singularities from the right half-plane ($\mathrm{Re}\,k>0$) of the sheet $\boldsymbol{C}_k$ are thought of.) From (19) it follows:

$$\begin{aligned} U(g,t) \approx\; 2\,\mathrm{Im}\Bigg\{&\sum_{n:\mathrm{Re}\,\overline{k}_n>0} u(g,\overline{k}_n)e^{-i\overline{k}_n t}\int\limits_{\boldsymbol{Q}} u(g_0,\overline{k}_n)\tilde{f}(g_0,\overline{k}_n)dg_0 \\ &+ e^{-i\underline{k}t}\int\limits_{\boldsymbol{Q}}\tilde{G}(g,g_0,\underline{k})f_{-1}(g_0,\underline{k})dg_0\Bigg\} \\ =\; 2\Bigg\{&\sum_{n:\mathrm{Re}\,\overline{k}_n>0} e^{t\,\mathrm{Im}\,\overline{k}_n}\left|u(g,\overline{k}_n)\right|\left|C(\tilde{f},\overline{k}_n)\right| \end{aligned}$$

$$\sin\left[\arg u(g,\overline{k}_n)+\arg C(\tilde{f},\overline{k}_n)-t\,\mathrm{Re}\,\overline{k}_n\right]$$
$$+e^{t\,\mathrm{Im}\,\underline{k}}\left|\tilde{U}(g,\underline{k},f_{-1})\right|\sin\left[\arg\tilde{U}(g,\underline{k},f_{-1})-t\,\mathrm{Re}\,\underline{k}\right]\Bigg\}. \quad (24)$$

Here the term is added, which represents the field oscillating with a frequency $\mathrm{Re}\,\underline{k}$. Its spatial configuration is determined by the solution $\tilde{U}(g,\underline{k},f_{-1})$ of the elliptical problem $A(\underline{k})[\tilde{U}(g,\underline{k},f_{-1})]=f_{-1}(g,\underline{k})$, while its amplitude decreases as $\exp(t\,\mathrm{Im}\,\underline{k})$. With $\mathrm{Im}\,\underline{k}=0$ and sufficiently great t, this term will dominate in the field $U(g,t)$: the so-called 'limiting amplitude principle' is realized.

C. The function $\tilde{f}(g,k)$ possesses one second-order pole at the point $k=\underline{k}$ that is not coincide with the members $\overline{k}$ of the set Ω_k. In this case we have:

$$\begin{aligned}U(g,t)\approx 2\,\mathrm{Im}\Bigg\{&\sum_{n:\mathrm{Re}\,\overline{k}_n>0}u(g,\overline{k}_n)e^{-i\overline{k}_nt}\int\limits_{\boldsymbol{Q}}u(g_0,\overline{k}_n)\tilde{f}(g_0,\overline{k}_n)dg_0\\
&-ite^{-i\underline{k}t}\int\limits_{\boldsymbol{Q}}\tilde{G}(g,g_0,\underline{k})f_{-2}(g_0,\underline{k})dg_0\\
&+e^{-i\underline{k}t}\int\limits_{\boldsymbol{Q}}\left[\tilde{G}(g,g_0,\underline{k})f_{-1}(g_0,\underline{k})+G_1(g,g_0,\underline{k})f_{-2}(g_0,\underline{k})\right]dg_0\Bigg\}\\
=2\Bigg\{&\sum_{n:\mathrm{Re}\,\overline{k}_n>0}e^{t\,\mathrm{Im}\,\overline{k}_n}\left|u(g,\overline{k}_n)\right|\left|C(\tilde{f},\overline{k}_n)\right|\\
&\sin\left[\arg u(g,\overline{k}_n)+\arg C(\tilde{f},\overline{k}_n)-t\,\mathrm{Re}\,\overline{k}_n\right]\\
&-te^{t\,\mathrm{Im}\,\underline{k}}\left|\tilde{U}(g,\underline{k},f_{-2})\right|\cos\left[\arg\tilde{U}(g,\underline{k},f_{-2})-t\,\mathrm{Re}\,\underline{k}\right]\\
&+e^{t\,\mathrm{Im}\,\underline{k}}\left|\tilde{U}(g,\underline{k},f_{-1})+\tilde{U}_1(g,\underline{k},f_{-2})\right|\\
&\sin\left[\arg\left[\tilde{U}(g,\underline{k},f_{-1})+\tilde{U}_1(g,\underline{k},f_{-2})\right]-t\,\mathrm{Re}\,\underline{k}\right]\Bigg\}. \quad (25)\end{aligned}$$

Here

$$\tilde{U}_l(g,\eta,f)=\int\limits_{\boldsymbol{Q}}G_l(g,g_0,\eta)f(g_0,\eta)dg_0.$$

With $\mathrm{Im}\,\underline{k}=0$ and sufficiently large t, the contribution of free oscillations associated with the complex eigen frequencies $\overline{k}$ into the

field $U(g,t)$ is negligible. The field oscillating with the frequency $k=\underline{k}$ will dominate, whose spatial structure is determined by the solution $\tilde{U}(g,\underline{k},f_{-2})$ of the elliptical problem $A(\underline{k})[\tilde{U}(g,\underline{k},f_{-2})]=f_{-2}(g,\underline{k})$ and whose amplitude grows proportionally with t.

D. The simple poles $k=\underline{k}$ and $k=\overline{k}$ of the functions $\tilde{f}(g,k)$ and $\tilde{G}(g,g_0,k)$ coincide ($\overline{k}=\underline{k}$). This yields that

$$
\begin{aligned}
U(g,t) \approx 2\,\mathrm{Im}\Bigg\{ & \sum_{n:\mathrm{Re}\,\overline{k}_n>0;\overline{k}_n\neq\overline{k}} u(g,\overline{k}_n)e^{-i\overline{k}_n t}\int\limits_{\boldsymbol{Q}} u(g_0,\overline{k}_n)\tilde{f}(g_0,\overline{k}_n)dg_0 \\
& -ite^{-i\overline{k}t}u(g,\overline{k})\int\limits_{\boldsymbol{Q}} u(g,\overline{k})f_{-1}(g_0,\overline{k})dg_0 \\
& +e^{-i\overline{k}t}u(g,\overline{k})\int\limits_{\boldsymbol{Q}} u(g_0,\overline{k})f_0(g_0,\overline{k})dg_0 \\
& +e^{-i\overline{k}t}\int\limits_{\boldsymbol{Q}} G_0(g,g_0,\overline{k})f_{-1}(g_0,\overline{k})dg_0\Bigg\} \\
= 2\Bigg\{ & \sum_{n:\mathrm{Re}\,\overline{k}_n>0;\overline{k}_n\neq\overline{k}} e^{t\,\mathrm{Im}\,\overline{k}_n}\left|u(g,\overline{k}_n)\right|\left|C(\tilde{f},\overline{k}_n)\right| \\
& \sin\left[\arg u(g,\overline{k}_n)+\arg C(\tilde{f},\overline{k}_n)-t\,\mathrm{Re}\,\overline{k}_n\right] \\
& -te^{t\,\mathrm{Im}\,\overline{k}}\left|u(g,\overline{k})\right|\left|C(f_{-1},\overline{k})\right|\cos\left[\arg u(g,\overline{k})+\arg C(f_{-1},\overline{k})-t\,\mathrm{Re}\,\overline{k}\right] \\
& +e^{t\,\mathrm{Im}\,\overline{k}}\left|u(g,\overline{k})\right|\left|C(f_0,\overline{k})\right|\sin\left[\arg u(g,\overline{k})+\arg C(f_0,\overline{k}_n)-t\,\mathrm{Re}\,\overline{k}\right] \\
& +e^{t\,\mathrm{Im}\,\overline{k}}\left|\tilde{U}_0(g,\overline{k},f_{-1})\right|\sin\left[\arg\tilde{U}_0(g,\overline{k},f_{-1})-t\,\mathrm{Re}\,\overline{k}\right]\Bigg\}. \qquad (26)
\end{aligned}
$$

The overlapping of the singularities of the Green function of problem (11), (13) and of the source function $\tilde{f}(g,k)$ results in a prevalence of the corresponding free oscillation in $U(g,t)$. It depends on the values $|\,\mathrm{Im}\,\overline{k}|$ and $|C(f_{-1},\overline{k})|$ how long the field

$$
\begin{aligned}
W(g,t) \;=\; & -2te^{t\,\mathrm{Im}\,\overline{k}}\left|u(g,\overline{k})\right|\left|C(f_{-1},\overline{k})\right| \\
& \cos\left[\arg u(g,\overline{k})+\arg C(f_{-1},\overline{k})-t\,\mathrm{Re}\,\overline{k}\right] \qquad (27)
\end{aligned}
$$

will remain dominant.

E. The simple poles $\overline{k}$ and $\underline{k}$ of the functions $\tilde{G}(g, g_0, k)$ and $\tilde{f}(g, k)$ do not coincide being closely spaced ($|\overline{k} - \underline{k}| \ll 1$). In this case,

$$\begin{aligned} U(g,t) \approx 2\Bigg\{ & \sum_{n:\operatorname{Re}\overline{k}_n>0;\overline{k}_n\neq\overline{k}} e^{t\operatorname{Im}\overline{k}_n} \left|u(g,\overline{k}_n)\right| \left|C(\tilde{f},\overline{k})\right| \\ & \sin\left[\arg u(g,\overline{k}_n) + \arg C(\tilde{f},\overline{k}_n) - t\operatorname{Re}\overline{k}_n\right] \\ & - \operatorname{Im}\Bigg[ite^{-i\overline{k}t} u(g,\overline{k}) \int_{\boldsymbol{Q}} u(g_0,\overline{k}) f_{-1}(g_0,\underline{k}) dg_0 \\ & -e^{-i\overline{k}t} u(g,\overline{k}) \int_{\boldsymbol{Q}} u(g_0,\overline{k}) f_0(g_0,\underline{k}) dg_0 \\ & - e^{-i\underline{k}t} \int_{\boldsymbol{Q}} G_0(g,g_0,\overline{k}) f_{-1}(g_0,\underline{k}) dg_0 \Bigg] \Bigg\} + O\left(t^2|\overline{k}-\underline{k}|\right). \end{aligned} \quad (28)$$

Representations (26) and (28) are almost identical. The only distinction is that the excitation level both of the dominating component

$$W(g,t) = -2te^{t\operatorname{Im}\overline{k}}\operatorname{Re}\left[e^{-it\operatorname{Re}\overline{k}} u(g,\overline{k}) \int_{\boldsymbol{Q}} u(g_0,\overline{k}) f_{-1}(g_0,\underline{k}) dg_0 \right] \quad (29)$$

and of the background components oscillating with the frequencies $\operatorname{Re}\overline{k}$ and $\operatorname{Re}\underline{k}$ is determined somewhat differently than in the above-mentioned situations.

6.2. Selection of Sources in Numerical Experiments

In numerical experiments, where the amplitude centers $\underline{k}$ of the exciting signals do not coincide exactly with the eigen frequencies $\overline{k}$ and the observation interval $[0, T]$ is not too long, the contributions of the quasi-monochromatic components into the field $U(g, t)$ are mostly commensurable. Consequently, the further analysis calls for variation of the source function $F(g, t)$ in such a way that a particular oscillation dominates against the background of the remaining ones.

Consider the situation described by (22). For an oscillation with the eigen frequency $\overline{k}$ to be isolated from free oscillations of comparable Q-factor, we need only to take the source $\tilde{f}(g, k)$ such

that $|C(\tilde{f},\overline{k})| \gg |C(\tilde{f},\overline{k}_n)|,\ \overline{k}_n \neq \overline{k}$. Formula (23) determines the requirements for the function $\tilde{f}(g,k)$:

- $\tilde{f}(g,k)$ as a function of k must have in the frequency range under study a single and sufficiently sharp amplitude center in the vicinity of the point $k=\overline{k}$;
- $\tilde{f}(g,\overline{k})$ as an element of the $\boldsymbol{L}_2(\boldsymbol{Q}_b)$-space must be as 'parallel' to the element of the same space $w(g,\overline{k}^*)=u^*(g,\overline{k})$ as possible.

Both of these requirements are easily satisfiable when $\tilde{f}(g,k) \leftrightarrow F(g,t)$ and some *a priori* information on a spatial structure of $u(g,\overline{k})$ is available.

The choice of the source parameters is illustrated by the following example. In Fig. 2 the results of numerical experiments with

$$F(g,t) = 10\chi[3.5-|y|]\chi[1.5-|z+1|]\cos\left(\beta_1\tilde{k}y+\beta_2\right)\cos\left(\beta_3\tilde{k}z+\beta_4\right) \times \exp\left[-(t-\tilde{T})^2/4\tilde{\alpha}^2\right]\cos[\tilde{k}(t-\tilde{T})]\chi(t-T) = P_1(g)F_1(t), \quad (30)$$

are presented. This source function has seven free parameters ($\tilde{k}$, $\tilde{\alpha}$, $\tilde{T}$, and $\beta_j,\ j=1,\ldots,4$). The parameter $\tilde{k}$ specifies an amplitude center of the primary signal in the spectral domain (Fig. 2a), namely, the point at which an absolute value of the function

$$\tilde{U}_{\text{prim}}(g,k,\tilde{f}) = \int_0^T U_{\text{prim}}(g,t)e^{ikt}dt \leftrightarrow \begin{cases} U_{\text{prim}}(g,t), & t \le T \\ 0, & t > T \end{cases} \quad (31)$$

is maximal. Here $U_{\text{prim}}(g,t)$ represents the field generated by $F(g,t)$ in free space. Together with $\tilde{\alpha}$, it determines a range $[\tilde{k}-b/\tilde{\alpha};\tilde{k}+b/\tilde{\alpha}]$ of real frequencies k, where the normilized spectral amplitudes of $U_{\text{prim}}(g,t)$ ($|\tilde{U}_{\text{prim}}(g,k,\tilde{f})|/|\tilde{U}_{\text{prim}}(g,\tilde{k},\tilde{f})|$) do not exceed γ. On the t-axis, the value $|U_{\text{prim}}(g,t)|/|U_{\text{prim}}(g,\tilde{T})|$ does not exceed γ out of the range $\tilde{T}-c\tilde{\alpha} \le t \le \tilde{T}+c\tilde{\alpha}$ occupied by the signal $U_{\text{prim}}(g,t)$. In Table 1 we give approximate values of b and c obtained [11] from the well-known analytical representations for some fixed γ.

Parameters β_j allows one to excite the oscillation of a given symmetry class.

The source parameters obviously depend on whether range characteristics or separate oscillations of an OR are of interest. When studying an OR in a frequency range (Fig. 2b), $\tilde{k}$ is made to agree with the range center, while $\tilde{\alpha}$ is chosen such that the level of the normalized spectral amplitudes of $U_{\text{prim}}(g,t)$ (of the function $\tilde{f}(g,k)$) is not too low

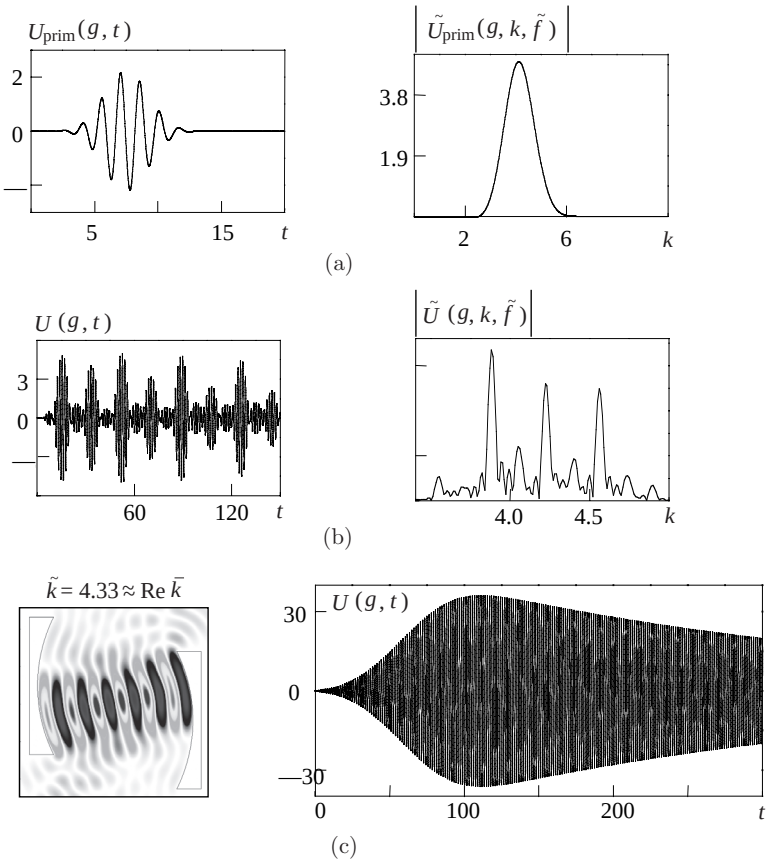

Figure 2. Temporal and spectral characteristics of the source (30) at $g = \{0, 0\}$ (a) and of the confocal resonator (36) at $g = \{0.2, 0.2\}$ (b); $\tilde{k} = 4.2,\ \tilde{\alpha} = 1,\ \tilde{T} = 6,\ \beta_1 = 1,\ \beta_2 = \pi/4,\ \beta_3 = \beta_4 = 0,\ T = 150$. (c) $H_{12,1}$-oscillation in the OR (34). The spatial field distribution and the field amplitude at $g = \{0.5, -1\};\ \tilde{k} = 4.33,\ \tilde{\alpha} = 20,\ \tilde{T} = 60,\ \beta_1 = 1,\ \beta_2 = \pi/4,\ \beta_3 = \beta_4 = 0,\ T = 300$.

(> 0.5 is desirable). To reduce the calculating time, the left boundary of the interval $\tilde{T} - c\tilde{\alpha} \le t \le \tilde{T} + c\tilde{\alpha}$ is placed at $t = 0$. In order to conserve the desired spectral characteristics of the source, the value of $|U_{\text{prim}}(g, 0)|$ must be negligible ($0.001 \le \gamma \le 0.01$). This requirement along with $\tilde{\alpha}$ determine the effective duration of the signal $0 \le t \le 2\tilde{T}$. When studying the frequency characteristics, oscillations belonging to a certain symmetry class are usually of interest. Therefore, the function

Table 1. Approximate values of b and c ensuring a given signal duration and frequency band.

	$\gamma = 0.001$	$\gamma = 0.01$	$\gamma = 0.1$	$\gamma = 0.5$
$b \approx$	2.63	2.14	1.52	0.83
$c \approx$	5.25	4.29	3.04	1.66

$\tilde{f}(g,k)$ must belong to the same symmetry class in its spatial structure and ensure much the same magnitudes of the coefficients $C(\tilde{f},\overline{k})$ (see formula (23)). Then the study of spectral characteristics of an OR in a frequency range reduces to the determination of the field $U(g,t)$ at a fixed point $g \in \boldsymbol{Q}_L$ as a function of $t \in [0,T]$ with a consequent analysis of its transform $\tilde{U}(g,k,\tilde{f}) \leftrightarrow U(g,t)$ (the function $U(g,t)$ is assumed zero outside of the interval $t \in [0,T]$).

In the study of separate oscillations $u(g,\overline{k})$ (Fig. 2c), the interval $[\tilde{k} - b/\tilde{\alpha}; \tilde{k} + b/\tilde{\alpha}]$ ($\tilde{k} \approx \operatorname{Re}\overline{k}$) may not contain the resonance points adjacent to $\operatorname{Re}\overline{k}$, while the spectral amplitudes of $U_{\text{prim}}(g,t)$ must be negligible at the ends of this interval. Clearly, this requirement can be weakened by considering the classes of symmetry of the analyzed and the adjacent oscillations.

The source

$$F(g,t) = P_1(g)\frac{\sin[\Delta k(t-\tilde{T})]}{(t-\tilde{T})}\cos[\tilde{k}(t-\tilde{T})]\chi(t-T) = P_1(g)F_2(t) \quad (32)$$

generates the signals $U_{\text{prim}}(g,t)$ with a more suitable distribution of spectral amplitudes as compared with (30) (Fig. 3). In the frequency range $[\tilde{k}-\Delta k; \tilde{k}+\Delta k]$, the absolute value of the function $\tilde{f}_2(k) \leftrightarrow F_2(t)$ remains practically constant, whereas outside of this interval we have $|\tilde{f}_2(k)| \approx 0$ for all $k > 0$.

The chief drawback of the sources (30) and (32) is the lack of information on the singularities of the function $\tilde{f}(g,k)$ in the lower half-plane of the sheet $\boldsymbol{C}_k$. The presence of poles $\underline{k}$, their orders and location can be inferred only indirectly by the behavior of the function $\tilde{f}(g,k)$ in the domain of real frequencies k. To tune more precisely for a certain singular point $\underline{k}$, one can use, for example [22], the sources $F(g,t)$ with the time function like

$$F_3(t) = \frac{1}{\operatorname{Re}\underline{k}}e^{t\operatorname{Im}\underline{k}}\sin(t\operatorname{Re}\underline{k}) \leftrightarrow \tilde{f}_3(k) = -\frac{1}{(k-\underline{k})(k+\underline{k}^*)}. \quad (33)$$

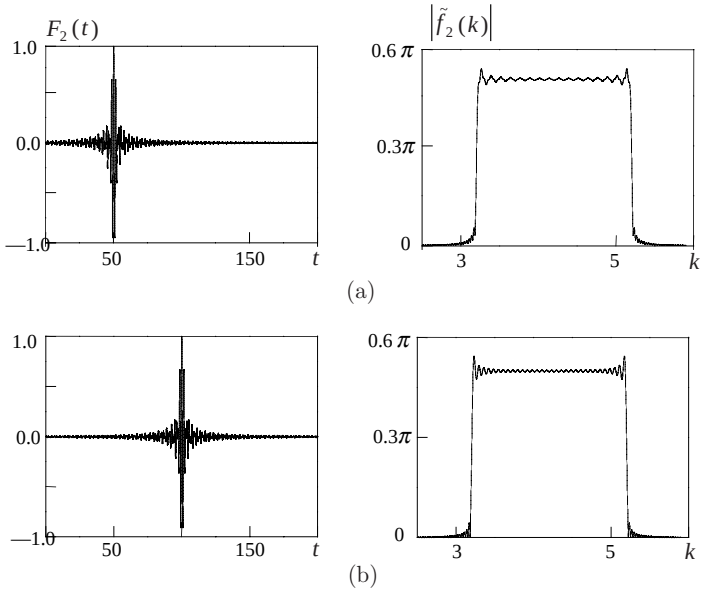

Figure 3. The characteristics of the source given by (32): $\tilde{k} = 4.2,\ \Delta k = 1,\ T = 200,\ \tilde{T} = 50$ (a) and $\tilde{T} = 100$ (b).

6.3. Analysis of the Results

In Fig. 2c the source (30) separates the $H_{12,1}$-oscillation from the spectrum of the following OR:

$$\sigma(g) = 2.19 \cdot 10^8 \chi[5 - |y|] \left\{ \chi[4 - |z|] \chi \left[z^2 + (y - 4.5)^2 - 9^2 \right] + \chi\left[4 - |z + D|\right] \chi \left[(z + D)^2 + (y + 4.5)^2 - 9^2 \right] \right\}. \quad (34)$$

Formula (34) describes the confocal copper resonator whose reflectors are shifted relative to each other by $D = 2.0$. All dimensions are given in centimeters. The subscripts m and n in the oscillation descriptor $H_{m,n}$ indicate the number of semi-variations of the field along y-axis and z-axis, respectively. The source is practically 'turned-off' at $t = 2\tilde{T} = 120$, and hence, for $\tau = t - 2\tilde{T} > 0$ we have from (22)

$$U(g,t) \approx U(\tau) = A \exp(\tau \operatorname{Im} \overline{k}) \cos(\tau \operatorname{Re} \overline{k} + a) \quad (35)$$

for every fixed point $g \in \boldsymbol{Q}_b$. By comparing (35) with $U(g,t)$ plotted in Fig. 2c, we obtain the following estimates: $\operatorname{Re} \overline{k} \approx \tilde{k} = 4.33,\ \operatorname{Im} \overline{k} \approx -0.0034,\ A \approx 37$, and $a \approx 0.79$.

In the simple case considered above, it has been possible to minimize the contribution of other free oscillations into the field $U(g,t)$ due to the absence of other eigen frequencies adjacent to $\overline{k}$. The analysis becomes more complicated if two eigen frequencies $\overline{k}_1$ and $\overline{k}_2$ are close to an extent that none of them can be separated even with a considerable bandwidth reduction of the signal $U_{\text{prim}}(g,t)$ in the spectral domain.

To illustrate this situation, consider (Fig. 4) the following confocal resonator

$$\sigma(g) = 2.19 \cdot 10^8 \chi[5-|y|]\chi[4-|z|]\chi\left[z^2+(|y|+4.5)^2-9^2\right] \qquad (36)$$

excited by the source (30) with $\tilde{k}=4.235$, $\tilde{\alpha}=50$, $\tilde{T}=150$, $\beta_1=1$, $\beta_2=0.785$, $\beta_3=\beta_4=0$, and $T=2500$. The spectral characteristics of the source as well as its spatial configuration are such that during the observation period two oscillation modes $H_{12,1}$ with $\operatorname{Re}\overline{k}_1\approx 4.2212$ and $H_{11,3}$ with $\operatorname{Re}\overline{k}_2\approx 4.239$ dominate concurrently in the field $U(g,t)$ (Fig. 4a). In Fig. 4b the function $U(g,t)$ at the point $g=\{0.82,0.0\}$ is plotted for $\tau=t-2\tilde{T}>0$.

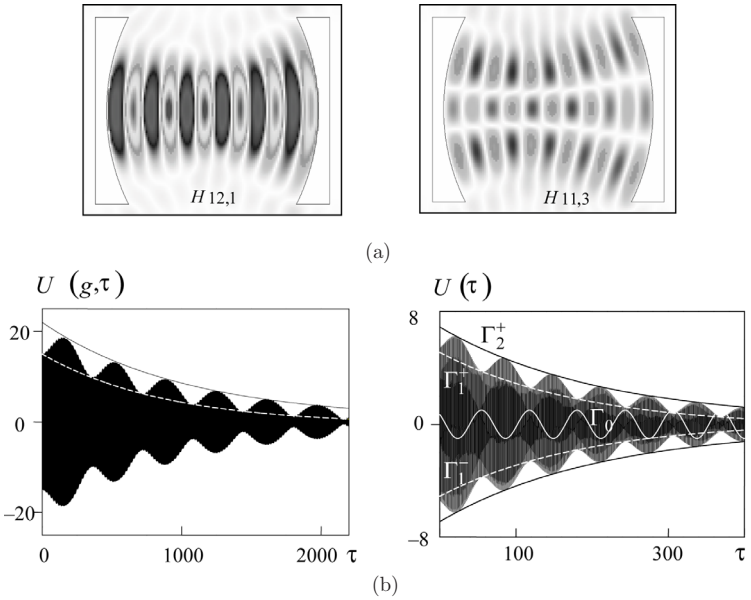

Figure 4. To the analysis of free oscillations with close eigen frequencies $\overline{k}_1$ and $\overline{k}_2$.

From (22) we obtain

$$\begin{aligned} U(g,t) &\approx U(\tau) = U_1(\tau) + U_2(\tau) \\ &= A\exp(\tau \operatorname{Im}\overline{k}_1)\cos(\tau \operatorname{Re}\overline{k}_1) \\ &\quad + B\exp(\tau \operatorname{Im}\overline{k}_2)\cos(\tau \operatorname{Re}\overline{k}_2 + b),\ \tau > 0. \end{aligned} \tag{37}$$

By introducing the notation $\Gamma_0(\tau) = \cos[\tau(\operatorname{Re}\overline{k}_2 - \operatorname{Re}\overline{k}_1) + b]$ we can verify (see a model example in Fig. 4c with $\operatorname{Re}\overline{k}_1 = 5.1$, $\operatorname{Re}\overline{k}_2 = 5.2$, $A = 0.9$, $\operatorname{Im}\overline{k}_1 = -0.002$, $B = -6.0$, $\operatorname{Im}\overline{k}_2 = -0.005$, $b = 0.8$) that:

- the curves
  $\Gamma_1^{\pm}(\tau) = \pm\left[|A|\exp(\tau \operatorname{Im}\overline{k}_1) - |B|\exp(\tau \operatorname{Im}\overline{k}_2)\right]$ and
  $\Gamma_2^{\pm}(\tau) = \pm\left[|A|\exp(\tau \operatorname{Im}\overline{k}_1) + |B|\exp(\tau \operatorname{Im}\overline{k}_2)\right]$
  represent 'global inner' and 'global outer' envelopes of $U(\tau)$;
- for $A > 0,\ B > 0$ at the points of contact of $U(\tau)$ and $\Gamma_2^{\pm}(\tau)$ we have:
  $\Gamma_0(\tau) = 1$ and $\cos(\tau \operatorname{Re}\overline{k}_1) = \cos(\tau \operatorname{Re}\overline{k}_2 + b) = \pm 1$,
  while at the points of contact of $U(\tau)$ and $\Gamma_1^{\pm}(\tau)$ we have:
  $\Gamma_0(\tau) = -1$ and $\cos(\tau \operatorname{Re}\overline{k}_1) = -\cos(\tau \operatorname{Re}\overline{k}_2 + b) = \pm 1$;
- for $A < 0,\ B < 0$ at the points of contact of $U(\tau)$ and $\Gamma_2^{\pm}(\tau)$ we have:
  $\Gamma_0(\tau) = 1$ and $\cos(\tau \operatorname{Re}\overline{k}_1) = \cos(\tau \operatorname{Re}\overline{k}_2 + b) = \mp 1$,
  while at the points of contact of $U(\tau)$ and $\Gamma_1^{\pm}(\tau)$ we have:
  $\Gamma_0(\tau) = -1$ and $\cos(\tau \operatorname{Re}\overline{k}_1) = -\cos(\tau \operatorname{Re}\overline{k}_2 + b) = \mp 1$,
- for $A > 0,\ B < 0$ at the points of contact of $U(\tau)$ and $\Gamma_1^{\pm}(\tau)$ we have:
  $\Gamma_0(\tau) = 1$ and $\cos(\tau \operatorname{Re}\overline{k}_1) = \cos(\tau \operatorname{Re}\overline{k}_2 + b) = \pm 1$,
  while at the points of contact of $U(\tau)$ and $\Gamma_2^{\pm}(\tau)$ we have:
  $\Gamma_0(\tau) = -1$ and $\cos(\tau \operatorname{Re}\overline{k}_1) = -\cos(\tau \operatorname{Re}\overline{k}_2 + b) = \pm 1$;
- for $A < 0,\ B > 0$ at the points of contact of $U(\tau)$ and $\Gamma_1^{\pm}(\tau)$ we have:
  $\Gamma_0(\tau) = 1$ and $\cos(\tau \operatorname{Re}\overline{k}_1) = \cos(\tau \operatorname{Re}\overline{k}_2 + b) = \mp 1$,
  while at the points of contact of $U(\tau)$ and $\Gamma_2^{\pm}(\tau)$ we have:
  $\Gamma_0(\tau) = -1$ and $\cos(\tau \operatorname{Re}\overline{k}_1) = -\cos(\tau \operatorname{Re}\overline{k}_2 + b) = \mp 1$.

The above facts allow one to determine uniquely basic parameters of free oscillations possessing close eigen frequencies $\overline{k}_1$ and $\overline{k}_2$ from the behavior of the function $U(\tau) \approx U(g,t)$ for $\tau > 0$. Thus, for example, for the case presented in Fig. 4a,b, we obtain: $\operatorname{Re}\overline{k}_2 - \operatorname{Re}\overline{k}_1 \approx 0.0178$, $A \approx 3.5$, $\operatorname{Im}\overline{k}_1 \approx -0.0005$, $B \approx 18.5$, $\operatorname{Im}\overline{k}_2 \approx -0.00105$, $b \approx 2.91$.

7. CONCLUSION

Until recently, it was customary to analyze spectral problems of the theory of OR in the context of FD methods. In the present work, the methodology has been developed for applying TD algorithms to the problems of this kind. This enables the range of rigorously solvable fundamental and applied problems to be considerably extended, since the TD methods operate on practically arbitrary geometry of electrodynamic objects allowing inclusion of any kind of metal-dielectric, magnetic, or plasma inhomogeneities. At the same time, the boundary conditions truncating the computation domain are exact, and hence, they do not distort the simulated processes.

The analytical and methodological results together with the associated software have made possible the detailed and reliable physical analysis of transient and steady-state processes in such open structures that are used in solid-state and vacuum electronics, resonant quasi-optics, and resonant antennas. The relevant results are to be presented in future papers.

APPENDIX A. TABLE OF SYMBOLS

- $\boldsymbol{R}^n$ and $\boldsymbol{G} \subset \boldsymbol{R}^n$ – the n-dimensional Euclidean space and the domain $\boldsymbol{G}$ in it.
- $\boldsymbol{L}_n(\boldsymbol{G})$ – the space of the functions $f(g),\ g \in \boldsymbol{G}$, such that a function $|f(g)|^n$ is integrable in $\boldsymbol{G}$.
- $\boldsymbol{W}^l_m(\boldsymbol{G})$ – the set of all elements $f(g)$ from $\boldsymbol{L}_m(\boldsymbol{G})$ that have generalized derivatives up to the order of l inclusive belonging to $\boldsymbol{L}_m(\boldsymbol{G})$.
- $\dot{\boldsymbol{W}}^1_2(\boldsymbol{G})$ – the subspace in $\boldsymbol{W}^1_2(\boldsymbol{G})$ such that $\boldsymbol{D}(\boldsymbol{G})$ is a dense set.
- $\boldsymbol{D}(\boldsymbol{G})$ – the set of finite infinitely differentiable in $\boldsymbol{G}$ functions.
- $\boldsymbol{L}_{2,1}(\boldsymbol{G}^T)$ – the space composed of all elements $f(g,t) \in \boldsymbol{L}_1(\boldsymbol{G}^T)$ with a finite norm $\|f\| = \int\limits_0^T \left(\int\limits_{\boldsymbol{G}} |f|^2 dg \right)^{1/2} dt$.
- $\boldsymbol{W}^1_{2,0}(\boldsymbol{Q}^T)$ – the subspace in $\boldsymbol{W}^1_2(\boldsymbol{G}^T)$ such that the smooth functions belonging to this subspace and going to zero in the vicinity of $\boldsymbol{P}^T = \boldsymbol{P} \times (0,T)$ form a dense set ($\boldsymbol{P}$ is the boundary of $\boldsymbol{G}$).
- l_2 – the space of infinite sequences $\left\{ a = \{a_n\} : \sum\limits_n |a_n|^2 < \infty \right\}$.

- $\chi[f_1(g)]\chi[f_2(g)]\cdots\chi[f_m(g)]$ – the generalized step-function such that it is equal to unity in the intersection $\boldsymbol{G}$ of the sets $\boldsymbol{G}_j = \{g \in \boldsymbol{R}^n : f_j(g) \geq 0\},\ j = 1, 2, \ldots, m$ and it is equal to zero in $\boldsymbol{R}^n \backslash \boldsymbol{G}$.

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