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Exact "Absorbing" Conditions in Initial-Boundary Value Problems in the Theory of Open Waveguide Resonators

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Abstract—Exact local and nonlocal radiation conditions are described for virtual boundaries in the cross-section of regular semi-infinite hollow waveguides, i.e., channels through which signals formed by resonance units propagate.

Keywords: computational electrodynamics, open waveguide resonator, initial-boundary value problem, exact absorbing conditions.

1. INTRODUCTION

The need to express our opinion concerning a very important problem of computational electrodynamics is a reaction to the following claim of the authors of paper [1]: "Therefore, it becomes clear that the boundary conditions are an integral part of a PDE (partial differential equation) problem and should always accompany the FDTD (finite-difference time-domain) formulation of it. This inflicts particular concerns when the problem under examination is a so-called 'open' space or unbounded problem, e.g., radiating, scattering, etc., meaning that the domain of interest is unbounded in one or more spatial-coordinate directions. For such problems, there are no exact boundary conditions known."

The fact is that exact absorbing boundary conditions (ABCs) that make it possible to efficiently restrict the computational space of finite difference methods and solve one of the main problems of computational electrodynamics and computational physics as a whole are available and have long been used thorough modeling and analysis of processes that are of interest both to applied and fundamental science. In paper [2], the exact nonlocal conditions for virtual boundaries in the cross section of regular semi indefinite hollow waveguides that conduct signals generated by a resonance unit were first published. Later, see, for example, [3–6], the approach suggested in [2], which is based on the use of radiation conditions for space–time amplitudes of the partial components (modes) of nonsinusoidal waves that propagate from the domain where effective sources and scatters are localized, was modified and extended to many other problems of theoretical and applied radiophysics. These are problems concerning antennas, analysis and synthesis of quasi-optical open dispersive resonators, wave propagation in a man-made environment, and so on. For some particular cases, nonlocality and corner point (i.e., points of intersection of virtual coordinate boundaries) problems are solved exactly. The efficiency and validity of the approaches based on the use of exact absorbing conditions is confirmed by problem-oriented numerical experiments and the solution of numerous test problems.

These questions are discussed in detail in [6]. In that study, the author considers concrete problems and constructs absorbing conditions taking into account the specific features of those problems and selecting optimal variants among numerous possible ones; the optimal variants of absorbing conditions are those that ensure the minimal approximation error while meeting stringent requirements for the computer resources. In this paper, we focus on the general theory and technical details concerning the construction of absorbing conditions. We also try to represent the results in a form that is convenient for solving various scalar and vector model initial-boundary value problems of the theory of open waveguide, periodic, and compact resonators, as well as a wide range of problems concerning the analysis of radiation processes and the propagation of nonsinusoidal and sinusoidal waves.

2. COMPACT WAVEGUIDE UNIT AND TWO-DIMENSIONAL SCALAR PROBLEMS IN THE CARTESIAN FRAME OF REFERENCE

2.1. Transformation of the Evolutionary Basis of the Signal in a Regular Parallel-Plate Waveguide

A waveguide unit (a waveguide transformer or an open waveguide resonator) is a classical model object in electrodynamics that may have diverse properties and a wide selection of material and geometric parameters. In this section, we consider the units in which the examination of the electromagnetic field is reduced to solving two-dimensional (in the plane $y0z$) initial-boundary value problems

$$\left[-\varepsilon(g)\frac{\partial^2}{\partial t^2}-\sigma(g)\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}\right]U(g,t) = F(g,t), \quad t>0, \quad g = \{y,z\}\in Q,$$

$$U(g,t)\big|_{t=0} = \varphi(g), \quad \frac{\partial}{\partial t}U(g,t)\Big|_{t=0} = \psi(g), \quad g\in\bar{Q}, \tag{2.1}$$

$$E_{tg}(g,t)\big|_{g\in S} = 0, \quad t\ge 0.$$

These are the units that have a constant cross section $x = \text{const}$ in any plane. An example is shown in Fig. 1 (a waveguide T-junction). When $U(g, t) = E_x$, we have $E_{tg}(g, t)|_{g\in S} = U(g, t)|_{g\in S}$, and problems (2.1) correspond to the case of E polarization of the field ($E_y = E_z = H_x = 0$). When $U(g, t) = H_x$ and $E_{tg}(g, t)|_{g\in S} = \partial U(g, t)/\partial\mathbf{n}|_{g\in S}$, where n is the normal to the contour S, and for piecewise constant $\varepsilon(g)$, $\sigma(g)$, the solutions of problems (2.1) determine H polarized fields ($H_y = H_z = E_x = 0$). Here, $E_{(\dots)}$ and $H_{(\dots)}$ are the components of the vectors of the intensity of the electric field (**E**) and the magnetic field (**H**); $\sigma = \eta_0\sigma_0$, $\varepsilon \equiv \varepsilon(g) \ge 1$ and $\sigma_0 \equiv \sigma_0(g) \ge 0$ are, respectively, the relative permittivity and the specific conductivity of the locally inhomogeneous (isotropic, nondispersive, and nonmagnetic) medium in which the waves propagate; $\eta_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the impedance of free space; ε_0 and μ_0 are the electric and magnetic constants of free space; t has the dimensionality of length (the product of the true time and the speed of light in free space); and $\varphi(g)$, $\psi(g)$, and $F(g, t)$ are the instantaneous and current sources of signals and pulse waves. The domain under analysis is the part of the plane R^2 bounded by the contour S, and $S \times [|x| \le \infty]$ is the surface of perfect conductors. It is assumed that the functions $F(g, t)$, $\varphi(g) = U^i(g, 0)$, and $\psi(g) = \partial U^i(g, t)/\partial t|_{t=0}$ ($U^i(g, t)$ is the incident wave) have a compact support in the closure of Q, and $\sigma(g)$ and $\varepsilon(g) - 1$ satisfy the assumptions of the theorem on the unique solvability of problem (2.1) in the energy class (see [7]). Their supports belong to the bounded set $\overline{Q_L}\setminus L$. In regular waveguides (in the domain ${}_LQ = Q\setminus(Q_L\cup L)$), in which the field generated by the unit can propagate infinitely far, there are no effective sources and scatterers. The virtual boundary L (it is denoted by dashed lines in Fig. 1) coincides with the cross section of such waveguides.

For definiteness, consider one (the vertical, $z > 0$) regular waveguide of the unit. Here, $\varepsilon(g) \equiv 1$, and $\sigma(g) = \varphi(g) = \psi(g) = F(g, t) \equiv 0$. Assuming that the perturbation $U(g, t)$ has not yet reached the boundary $z = 0$ by the time $t = 0$, we separate the variables to obtain the following representation of the solution $U(g, t)$ of problem (2.1):

$$U(g,t) = \sum_n w_n(z,t)\mu_n(y), \quad z\ge 0, \quad 0\le y\le a, \quad t\ge 0, \quad n\in\{n\}. \tag{2.2}$$

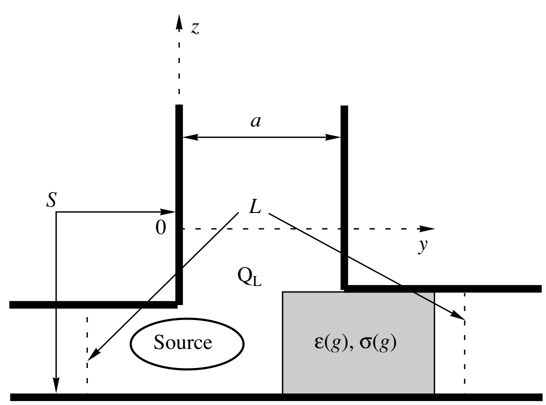

Fig. 1.

Series (2.2) with

$$w_n(z, t) = \langle U(g, t), \mu_n(y)\rangle \equiv \int_0^a U(g, t)\mu_n(y)dy \tag{2.3}$$

converges in the norm of $L_2(0; a)$ for any $z \geq 0$ and $t \geq 0$. The orthonormal system of transverse functions $\{\mu_n(y)\}$ is complete in the space $L_2(0; a)$ of square summable functions $f(y)$ such that $f(0) = f(a) = 0$ or $df(y)/dy|_{y=0,a} = 0$. This system is determined by the nontrivial solutions of the homogeneous (eigenvalue) problems

$$\left[\frac{d^2}{dy^2} + \lambda_n^2\right]\mu_n(y) = 0, \quad 0 < y < a,$$
$$\mu_n(0) = \mu_n(a) = 0 \ (E\text{-case}) \ \text{ or } \ d\mu_n(y)/dy|_{y=0,a} = 0 \ (H\text{-case}). \tag{2.4}$$

The space-time amplitudes $\{w_n(z, t)\}$ (the evolutionary bases of signals $U(g, t)$) are determined by the solutions of the initial-boundary value problems

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - \lambda_n^2\right]w_n(z, t) = 0, \quad t > 0,$$
$$w_n(z, 0) = 0, \quad \frac{\partial}{\partial t}w_n(z, t)\bigg|_{t=0} = 0, \tag{2.5}$$

where $z \geq 0$ and $n \in \{n\}$.

In the case of E polarization of the field, we have $\{n\} = 1, 2, \ldots, \mu_n(y) = \sqrt{2/a}\sin(n\pi y/a)$, and $\lambda_n = n\pi/a$. In the H case, $\{n\} = 0, 1, \ldots, \mu_n(y) = \sqrt{(2 - \delta_0^n)/a}\cos(n\pi y/a)$, and $\lambda_n = n\pi/a$.

The use of the cosine Fourier transform in (2.5) with respect to z on the semiaxis $z \geq 0$ (transform $\longleftrightarrow$ inverse transform)

$$\tilde{f}(\omega) = F_c[f](\omega) \equiv \sqrt{\frac{2}{\pi}}\int_0^\infty f(z)\cos(\omega z)dz \longleftrightarrow f(z) = F_c^{-1}[\tilde{f}](z) \equiv \sqrt{\frac{2}{\pi}}\int_0^\infty \tilde{f}(\omega)\cos(\omega z)d\omega \tag{2.6}$$

gives the following Cauchy problems for the transforms $\tilde{w}_n(\omega, t)$:

$$D(\sqrt{\lambda_n^2 + \omega^2})[\tilde{w}_n(\omega, t)] \equiv \left[\frac{\partial^2}{\partial t^2} + (\lambda_n^2 + \omega^2)\right]\tilde{w}_n(\omega, t) = -\sqrt{\frac{2}{\pi}}\frac{\partial w_n(z, t)}{\partial z}\bigg|_{z=0} = -\sqrt{\frac{2}{\pi}}w'_n(0, t),$$
$$\omega > 0, \quad t > 0, \quad \tilde{w}_n(\omega, 0) = 0, \quad \frac{\partial}{\partial t}\tilde{w}_n(\omega, t)\bigg|_{t=0} = 0, \quad \omega \geq 0. \tag{2.7}$$

In the derivation of (2.7), we used the fact that

$$-\omega^2\tilde{f}(\omega) - \sqrt{\frac{2}{\pi}}\left[\frac{d}{dz}f(z)\right]\bigg|_{z=0} \longleftrightarrow \frac{d^2}{dz^2}f(z)$$

and that the wave $U(z, t)$ in the part of the domain ${}_LQ$ under examination does not contain components that propagate in the direction of decreasing z. The components propagating in the direction $z = \infty$ vanish at every finite time $t = T$ for sufficiently large values of z.

Setting the functions $\tilde{w}_n(\omega, t)$ to zero on the semiaxis $t < 0$, we obtain the generalized formulation of the Cauchy problem (2.7) (see [8])

$$D(\sqrt{\lambda_n^2 + \omega^2})[\tilde{w}_n(\omega, t)] = -\sqrt{\frac{2}{\pi}}w'_n(0, t), \quad w > 0, \quad -\infty < t < \infty. \tag{2.8}$$

The convolution of the fundamental solution $G(\lambda, t) = \chi(t)\lambda^{-1}\sin(\lambda t)$ of the operator $D(\lambda)$ with the right-hand side of Eq. (2.8) allows us to write $\tilde{w}_n(\omega, t)$ in the form

$$\tilde{w}_n(\omega, t) = -\sqrt{\frac{2}{\pi}}\int_0^t \sin\left[(t-\tau)\sqrt{\lambda_n^2+\omega^2}\right]\frac{w_n'(0,\tau)}{\sqrt{\lambda_n^2+\omega^2}}d\tau, \quad \omega \geq 0, \quad t \geq 0. \tag{2.9}$$

Passing in (2.9) to the inverse transforms $\tilde{w}_n(\omega, t)$ by applying the inverse cosine Fourier transform (2.6), we obtain the representation

$$w_n(z, t) = -\int_0 J_0\{\lambda_n[(t-\tau)^2 - z^2]^{1/2}\}\chi[(t-\tau)-z]w_n'(0,\tau)d\tau, \quad z \geq 0, \quad t \geq 0, \tag{2.10}$$

where J_n are the cylindrical Bessel functions.

Equations (2.10) represent the general property of the solutions $U(g, t)$ of problems (2.1) in the domain $_LQ$ of solutions satisfying the zero initial conditions that do not contain components (modes) propagating in the direction of the compact inhomogeneity (unit). These equations specify the diagonal transport operator $Z_{0\to z}(t)$ acting by the rule

$$w(z, t) = \{w_n(z, t)\} = Z_{0\to z}(t)[w'(0,\tau)], \quad w'(b,\tau) = \{w_n'(b,\tau)\}, \quad z \geq 0, \quad t \geq \tau \geq 0$$

(see [5, 6, 9, 10]). This operator makes it possible to keep track of the change of field when a nonsinusoidal wave propagates without obstacles in a finite segment of a regular waveguide. Obviously, Eq. (2.10) can be also written in the form

$$w_n(z, t) = -\int_0 J_0\{\lambda_n[(t-\tau)^2 - (z-z_0)^2]^{1/2}\}\chi[(t-\tau)-(z-z_0)]w_n'(z_0,\tau)d\tau, \quad z \geq z_0 \geq 0, \quad t \geq 0. \tag{2.11}$$

2.2. Nonlocal Absorbing Conditions

Projecting the observation point in (2.10) on the imaginary boundary L ($z = 0$), we obtain

$$w_n(0, t) = -\int_0 J_0[\lambda_n(t-\tau)]\chi(t-\tau)w_n'(0,\tau)d\tau, \quad t \geq 0. \tag{2.12}$$

Differentiating Eq. (2.12) with respect to t, we obtain

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)w_n(z, t)\bigg|_{z=0} = \lambda_n\int_0 J_1[\lambda_n(t-\tau)]\chi(t-\tau)w_n'(0,\tau)d\tau, \quad t \geq 0. \tag{2.13}$$

Here, we used the well-known relations $dJ_0(x)/dx = -J_1(x)$, $J_0(0) = 1$, and $\chi^{(1)}(t-\tau) = \delta(t-\tau)$; δ is the Dirac delta function, and $\chi^{(1)}$ is the generalized derivative of the Heaviside step function χ.

Apply the Laplace transform with respect to t to (2.13) to obtain

$$\tilde{f}(s) = L[f](s) \equiv \int_0^\infty f(t)e^{-st}dt \longleftrightarrow f(t) = L^{-1}[\tilde{f}](t) \equiv \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty}\tilde{f}(s)e^{st}ds. \tag{2.14}$$

Taking into account the well-known relations

$$\tilde{f}_1(s)\tilde{f}_2(s) \longleftrightarrow \int_0^t f_1(t-\tau)f_2(\tau)d\tau$$

(the convolution theorem), $\lambda^2[\sqrt{s^2+\lambda^2}(\sqrt{s^2+\lambda^2}+s)]^{-1} \longleftrightarrow \lambda J_1(\lambda t)$, we obtain the following representation in the space of Laplace transforms $\tilde{w}_n(z, s)$:

$$\left(\frac{\partial}{\partial z} + s\right)\tilde{w}_n(z, s)\bigg|_{z=0} = \frac{\lambda_n^2\tilde{w}_n'(0, s)}{\sqrt{s^2+\lambda^2}(\sqrt{s^2+\lambda^2}+s)}. \tag{2.15}$$

Rewrite (2.15) in the form

$$\tilde{w}'_n(0, s) = -\left(s + \frac{\lambda_n^2}{s + \sqrt{s^2 + \lambda_n^2}}\right)\tilde{w}_n(0, s). \tag{2.16}$$

Returning in (2.16) to the inverse transforms and taking into account the relation $(s + \sqrt{s^2 + \lambda^2})^{-1} \longleftrightarrow (\lambda t)^{-1} J_1(\lambda t)$ (see [12]), we obtain

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) w_n(z, t)\Big|_{z=0} = -\lambda_n \int_0 J_1[\lambda_n(t-\tau)](t-\tau)^{-1}\chi(t-\tau) w_n(0, \tau) d\tau, \quad t \geq 0. \tag{2.17}$$

The justification of manipulations (2.13)–(2.17) is based on the estimates obtained in [13]. More precisely, at the points g of any bounded subdomain of Q, the field $U(g, t)$ produced by a system of sources with compact supports cannot grow more rapidly than $\exp(\alpha t)$ as $t \longrightarrow \infty$, where $\alpha > 0$ is a constant. These estimates hold for all electrodynamic structures whose spectrum of complex eigenfrequencies $\{k_n\}$ does not contain the points k_m with $\mathrm{Im} k_m > 0$ (see [14]). In particular, all the open waveguide resonators considered in this paper belong to this type of electrodynamic structure.

Using representations (2.2) and (2.3), rewrite Eqs. (2.12), (2.13), and (2.17) in the form

$$U(y, 0, t) = -\sum_n \left\{ \int_0^t J_0[\lambda_n(t-\tau)] \left[\frac{\partial}{\partial z}\langle U(\tilde{y}, z, \tau), \mu_n(\tilde{y})\rangle\right]\Big|_{z=0} d\tau \right\} \mu_n(y) = V_1(y, t), \tag{2.18}$$

$$0 \leq y \leq a, \quad t \geq 0,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) U(y, z, t)\Big|_{z=0} = \sum_n \left\{ \int_0^t J_1[\lambda_n(t-\tau)] \left[\frac{\partial}{\partial z}\langle U(\tilde{y}, z, \tau), \mu_n(\tilde{y})\rangle\right]\Big|_{z=0} d\tau \right\} \lambda_n \mu_n(y) = V_2(y, t), \tag{2.19}$$

$$0 \leq y \leq a, \quad t \geq 0,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) U(y, z, t)\Big|_{z=0} = -\sum_n \left\{ \int_0^t J_1[\lambda_n(t-\tau)](t-\tau)^{-1} \langle U(\tilde{y}, 0, \tau), \mu_n(\tilde{y})\rangle d\tau \right\} \lambda_n \mu_n(y) = V_3(y, t), \tag{2.20}$$

$$0 \leq y \leq a, \quad t \geq 0.$$

Returning to the main subject of this paper, we give a preliminary assessment of the possibility of using relations (2.18)–(2.20) as conditions that restrict the theoretically open domain of wave propagation in the process being simulated.

First, problems (2.1) and (2.1) in combination with any of the conditions (2.18)–(2.20) are equivalent. This follows from the following three facts: the initial problem has a unique solution (see [7]), the solution of the initial problem is also a solution of the modified problem (by construction), and the modified problem has a unique solution. The last assertion can be proved by the conventional method based on the use of energy bounds on solutions $U(g, t)$ (bounds on U^2, $(\partial U/t)^2$ and $|\mathrm{grad}\, U|^2$). In [7, 8, 15], such proofs are given for the initial-boundary value problems with "ordinary" boundary conditions; and, in [2], a uniqueness theorem for the problem subject to condition (2.18) is proved.

Second, conditions (2.18)–(2.20) are exact; therefore, imposing them on the initial problems, we do not enhance the error of computations and do not distort the process being simulated.

Third, when discretizing problems (2.1) subject to conditions (2.18)–(2.20), we can totally eliminate the instantaneous effect of the function $U(g, \tau)$ that appears on the right-hand side of (2.18)–(2.20) (see [5, 6]). Therefore, when moving across time layers in the process of computation, the functions $V_j(y, t)$ may be assumed to be known (determined on the preceding layers) for the times τ that are strictly less than t.

Fourth, conditions (2.18)–(2.20) are correctly included in the standard finite difference scheme; i.e., they do not violate its stability and almost do not complicate its numerical implementation. A formal proof of this fact is very tedious, but it is based on the equivalence of the initial and modified problems and well-known classical results (see [7]).

Relations (2.12), (2.13), (2.17), and (2.18)–(2.20) provide an exact form of the radiation conditions for the outgoing nonsinosoidal waves produced by the unit, the conditions imposed on the space–time amplitudes of each partial component (mode) of the wave directed by the regular structure (see (2.12), (2.13), (2.17)), the conditions imposed on the field of such a wave as a whole (see (2.18)–(2.20)), and the conditions applied on the virtual boundaries coinciding with the cross section of the waveguide channels that are nonlocal both with respect to the spatial (y) and time (t) variable. The first of these conditions ((2.12) and (2.18)) were used as the exact ABCs in the pioneering paper [2].

In the statement of problems (2.1) and the definition of the domains Q and Q_L, we assumed that the functions of sources exciting the unit have a compact support in the closure of Q and their supports belong to $\bar{Q}_L \backslash L$ for all instances $0 \le t \le T$. We will also use similar assumptions below. Therefore, we will be able to formulate conditions on the virtual boundaries L in terms of the total field $U(g, t)$. It is clear that the restrictions due to this assumption can be completely or in part removed assuming that there is an incident (primary) wave $U^i(g, t)$. Then, the function $U(g, t)$ in all the representations must be replaced with the function $U^s(g, t) = U(g, t) - U^i(g, t)$, which describes the scattered (secondary) field. This simple procedure enables us to avoid an unpractical extension of the computational space for long primary signals.

Let us point out one more important fact. If the right-hand sides $V_j(y, t)$ in conditions (2.19) an (2.20) are replaced with zero, these conditions coincide with the simplest classical ABC that ensures the first order of approximation with respect to the angle of arrival of the plane complex wave [16, 17] (this simulates the absence of reflection). Therefore, the functions $V_j(y, t)$ may be used for making external estimates of the accuracy of the approaches that use the corresponding approximate absorbing conditions in their computational schemes. Estimates of this type can also be obtained for some other available ABCs.

2.3. Local Absorbing Conditions

The direct implementation of conditions (2.18)–(2.20) (based on the corresponding discrete analogs) places high requirements upon computer memory for storing the values of $V_j(y, t)$ (see Section 5). The size of the arrays increases at each time step, and all their elements are needed to compute V_1, V_2, and V_3 (a consequence of nonlocality). Various variants of the solution to this problem based on the analytic separation of variables in the arguments of the Bessel functions make it possible to reduce the dimension of the arrays by unity at the expense of a small increase of the number of operations (see [5, 6]). The following technique, which is easy to implement, enables us to replace the nonlocal conditions by local ones. Using the representation (see [18])

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\varphi)d\varphi,$$

we rewrite (2.12) in the form

$$w_n(0, t) = -\frac{2}{\pi}\int_0^{\pi/2}\left\{\int_0 \cos[\lambda_n(t-\tau)\sin\varphi]\chi(t-\tau)w_n'(0, \tau)d\tau\right\}d\varphi, \quad t \ge 0. \tag{2.21}$$

Define

$$u_n(t, \varphi) = -\int_0 \frac{\sin[\lambda_n(t-\tau)\sin\varphi]\chi(t-\tau)w_n'(0, \tau)}{\lambda_n\sin\varphi}d\tau, \quad t \ge 0, \quad 0 \le \varphi \le \pi/2. \tag{2.22}$$

Then,

$$\frac{\partial u_n(t, \varphi)}{\partial t} = -\int_0 \cos[\lambda_n(t-\tau)\sin\varphi]\chi(t-\tau)w_n'(0, \tau)d\tau$$

and (2.21) yields

$$w_n(0, t) = \frac{2}{\pi}\int_0^{\pi/2}\frac{\partial u_n(t, \varphi)}{\partial t}d\varphi, \quad t \ge 0. \tag{2.23}$$

The integral form (2.22) is equivalent to the differential form

$$\left(\frac{\partial^2}{\partial t^2}+\lambda_n^2\sin^2\varphi\right)u_n(t,\varphi) = -w_n'(0,t),\quad t>0,$$
$$u_n(0,\varphi) = \left.\frac{\partial u_n(t,\varphi)}{\partial t}\right|_{t=0} = 0. \tag{2.24}$$

Indeed, if we use the generalized statement of the corresponding Cauchy problem in (2.24) and use the fundamental solution $G(\lambda, t) = \chi(t)\lambda^{-1}\sin(\lambda t)$ of the operator $D(\lambda) \equiv [d^2/dt^2 + \lambda^2]$, we easily verify that (2.22) and (2.24) define the same function $u_n(t, \varphi)$.

Multiply (2.23) and (2.24) by $\mu_n(y)$ and sum over all $n \in \{n\}$. As a result, using the fact that

$$\sum_n\lambda_n^2u_n(t,\varphi)\mu_n(y) = -\partial^2W(y,t,\varphi)/\partial y^2 \text{ for } W(y,t,\varphi) = \sum_n u_n(t,\varphi)\mu_n(y)$$

(see (2.4)), we obtain

$$U(y,0,t) = \frac{2}{\pi}\int_0^{\pi/2}\frac{\partial W(y,t,\varphi)}{\partial t}d\varphi,\quad t\ge 0,\quad 0\le y\le a,$$
$$\left(\frac{\partial^2}{\partial t^2}-\sin^2\varphi\frac{\partial^2}{\partial y^2}\right)W(y,t,\varphi) = -\left.\frac{\partial U(y,z,t)}{\partial z}\right|_{z=0},\quad 0<y<a,\quad t>0, \tag{2.25}$$
$$W(y,0,\varphi) = \left.\frac{\partial W(y,t,\varphi)}{\partial t}\right|_{t=0} = 0,\quad 0\le y\le a,$$
$$W(0,t,\varphi) = W(a,t,\varphi) = 0\ (E\text{-case})\ \text{ or }\ \partial W(y,t,\varphi)/\partial y|_{y=0,a} = 0\ (H\text{-case}),\quad t\ge 0.$$

This is an exact local (with respect to both the spatial and the time variables) absorbing condition that enables us to efficiently restrict the computational space when numerically solving problem (2.1). Here and in what follows, $W(y, t, \varphi)$ is an auxiliary function that is determined by solving a special (internal with respect to the corresponding condition) initial-boundary value problem, and $0 \le \varphi \le \pi/2$ is a numerical parameter.

Implementing the scheme described above and using relations (2.13) and (2.17) as the basic ones, we arrive at the following exact local ABCs, which are slightly different from (2.25):

$$\left.\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)U(y,z,t)\right|_{z=0} = \frac{2}{\pi}\int_0^{\pi/2}W(y,t,\varphi)\cos^2\varphi d\varphi,\quad t\ge 0,\quad 0\le y\le a,$$
$$\left(\frac{\partial^2}{\partial t^2}-\cos^2\varphi\frac{\partial^2}{\partial y^2}\right)W(y,t,\varphi) = -\frac{\partial^2}{\partial y^2}\left[\left.\frac{\partial}{\partial z}U(y,z,t)\right|_{z=0}\right],\quad 0<y<a,\quad t>0, \tag{2.26}$$
$$W(y,0,\varphi) = \left.\frac{\partial W(y,t,\varphi)}{\partial t}\right|_{t=0} = 0,\quad 0\le y\le a,$$
$$W(0,t,\varphi) = W(a,t,\varphi) = 0\ (E\text{-case})\ \text{ or }\ \partial W(y,t,\varphi)/\partial y|_{y=0,a} = 0\ (H\text{-case}),\quad t\ge 0;$$

$$\left.\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)U(y,z,t)\right|_{z=0} = \frac{2}{\pi}\int_0^{\pi/2}\frac{\partial W(y,t,\varphi)}{\partial t}\sin^2\varphi d\varphi,\quad t\ge 0,\quad 0\le y\le a,$$
$$\left(\frac{\partial^2}{\partial t^2}-\cos^2\varphi\frac{\partial^2}{\partial y^2}\right)W(y,t,\varphi) = \frac{\partial^2U(y,0,t)}{\partial y^2},\quad 0<y<a,\quad t>0, \tag{2.27}$$
$$W(y,0,\varphi) = \left.\frac{\partial W(y,t,\varphi)}{\partial t}\right|_{t=0} = 0,\quad 0\le y\le a,$$

$$W(0, t, \varphi) = W(a, t, \varphi) = 0 \;\; (E\text{-case}) \;\; \text{or} \;\; \partial W(y, t, \varphi)/\partial y|_{y = 0, a} = 0 \;\; (H\text{-case}), \quad t \geq 0.$$

When changing from (2.13) to (2.26), we used the representations

$$J_1(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\cos\varphi)\cos\varphi d\varphi \;\; \text{(see [19])},$$

$$u_n(t, \varphi) = \lambda_n \int_0 \frac{\sin[\lambda_n(t-\tau)\cos\varphi]\chi(t-\tau)w_n'(0, \tau)}{\cos\varphi} d\tau, \quad t \geq 0, \quad 0 \leq \varphi \leq \pi/2.$$

When changing from (2.17) to (2.27), we used the representations

$$J_1(x) = \frac{2x}{\pi}\int_0^{\pi/2} \cos(x\cos\varphi)\sin^2\varphi d\varphi$$

(the integral Poisson formula [18]),

$$u_n(t, \varphi) = -\lambda_n \int_0 \frac{\sin[\lambda_n(t-\tau)\cos\varphi]\chi(t-\tau)w_n(0, \tau)}{\cos\varphi} d\tau, \quad t \geq 0, \quad 0 \leq \varphi \leq \pi/2.$$

The assumption $W(y, t, \varphi) \equiv 0$ (it is almost impossible to justify) reduces (2.26), (2.27) to the classical ABC of the first order of approximation (see [16]). Replacing the integral in (2.26) by a finite sum by the trapezoid rule, we obtain an approximate condition that coincides (in its form) with conditions (17), (18) in [20].

3. AXIALLY SYMMETRIC WAVEGUIDE UNITS AND TWO-DIMENSIONAL SCALAR PROBLEMS IN THE CYLINDRICAL FRAME OF REFERENCE

3.1. Statement and General Solutions of Model Initial-Boundary Value Problems

Examination of TE_0 ($\partial/\partial\phi \equiv 0$, $E_\rho = E_z = H_\phi = 0$, see [5, 6]) and TM_0 ($H_\rho = H_z = E_\phi = 0$) waves in axially symmetric structures (for example, see Fig. 2, where a resonator in the break of a circular coaxial waveguide is depicted) is reduced to the solution of the following two-dimensional (in the half-plane of the variable $g = \{\rho \geq 0, z\}$) scalar initial-boundary value problems:

$$\left[-\varepsilon(g)\frac{\partial^2}{\partial t^2} - \sigma(g)\frac{\partial}{\partial t} + \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\right)\right]U(g, t) = F(g, t), \quad t > 0, \quad g \in Q, \tag{3.1a}$$

$$U(g, t)|_{t=0} = \varphi(g), \quad \left.\frac{\partial}{\partial t}U(g, t)\right|_{t=0} = \psi(g), \quad g \in \bar{Q}, \tag{3.1b}$$

$$E_{tg}(g, t)|_{g \in S} = 0, \quad U(0, z, t) = 0, \quad t \geq 0. \tag{3.1c}$$

The last condition in (3.1c) is caused by the symmetry of the problem: the axis $\rho = 0$ coincides with the circular symmetry axis; therefore, only the E_z and H_z components of the field can be distinct from zero. When $U(g, t) = E_\phi$, the equality $E_{tg}(g, t)|_{g \in S} = U(g, t)|_{g \in S}$ and problems (3.1) describe space–time transformations of TE_0 waves; when $\varepsilon(g)$ and $\sigma(g)$ are piecewise constant and $U(g, t) = H_\phi$, they describe transfor-

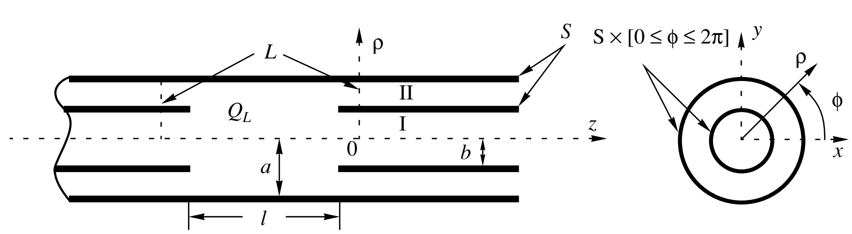

Fig. 2.

mations of TM_0 waves. The domain of interest Q is the part of the half plane $\rho 0z$ bounded by the contour S, and $S \times [0 \le \phi \le 2\pi]$ is the surface of perfect conductors. As before, it is assumed that the functions $F(g, t)$, $\varphi(g) = U^i(g, 0)$, $\psi(g) = \partial U^i(g, t)/\partial t|_{t=0}$ ($U^i(g, t)$ is the incident wave), $\sigma(g)$, and $\varepsilon(g) - 1$ have a compact support in the closure of Q and satisfy the assumptions of the unique solvability theorem for problems (3.1) in the energy class. The supports of these functions belong to the set $\bar{Q}_L \backslash L$. In regular circular and coaxial circular waveguides (in $_LQ = Q \backslash (Q_L \cup L)$) in which the field induced by the unit can propagate infinitely far, there are no sources and effective scatterers. The virtual boundary L (it is shown by the dashed line in Fig. 2) belongs to the plane of the cross section of these waveguides.

The geometry of the domain $_LQ$ in problems (3.1) is such that the general solutions of these problems in the corresponding regular partial domains (it is clear that it is sufficient to consider domains I and II with $z > 0$) can be represented in the form

$$U(z, \rho, t) = \sum_n w_n(z, t)\mu_n(\rho), \quad t \ge 0,$$
$$w_n(z, t) = \langle U(z, \rho, t), \mu_n(\rho)\rangle \equiv \int_{\rho_1}^{\rho_2} U(z, \rho, t)\mu_n(\rho)\rho d\rho, \tag{3.2}$$

where the orthonormal bases $\{\mu_n(\rho)\}$ ($n \in \{n\}$) are given by the nontrivial solutions of the homogeneous (eigenvalue) problems

$$\left(\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho + \lambda_n^2\right)\mu_n(\rho) = 0, \quad \rho \in (\rho_1 = 0, \rho_2 = b),$$
$$\mu_n(0) = \mu_n(b) = 0 \ \ (TE_0 \text{ waves}) \ \text{ or } \ \mu_n(0) = \left.\frac{d(\rho\mu_n(\rho))}{d\rho}\right|_{\rho = b} = 0 \ \ (TM_0 \text{ waves}) \tag{3.3}$$

(for domain I corresponding to a circular waveguide). For domain II corresponding to a circular coaxial waveguide, these bases are given by the nontrivial solutions of the problems

$$\left(\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho + \lambda_n^2\right)\mu_n(\rho) = 0, \quad \rho \in (\rho_1 = b, \rho_2 = a),$$
$$\mu_n(b) = \mu_n(a) = 0 \ \ (TE_0) \ \text{ or } \ \left.\frac{d(\rho\mu_n(\rho))}{d\rho}\right|_{\rho = b} = \left.\frac{d(\rho\mu_n(\rho))}{d\rho}\right|_{\rho = a} = 0 \ \ (TM_0). \tag{3.4}$$

The boundary values in problems (3.3) and (3.4), which are related to electric waves (TM_0 waves), were formulated with regard for the fact that, in the case under examination (see [5, 6]), $U(g, t) = H_\phi$, $E_{tg}(g, t)|_{g \in S} = E_z(g, t)|_{g \in S}$, and $\partial E_z/\partial t = (\eta_0/\rho)[\partial(\rho H_\phi)/\partial\rho]$.

The space–time amplitudes $w_n(z, t)$ (elements of the evolutionary bases of signals) are obtained by solving the initial-boundary value problems

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - \lambda_n^2\right)w_n(z, t) = 0, \quad t > 0,$$
$$w_n(z, 0) = 0, \quad \left.\frac{\partial}{\partial t}w_n(z, t)\right|_{t = 0} = 0, \tag{3.5}$$

where $z \ge 0$ and $n \in \{n\}$ (it is assumed that at $t = 0$ the perturbation $U(g, t)$ induced by the sources $\varphi(g)$, $\psi(g)$, and $F(g, t)$ concentrated in Q_L has not yet reached the boundary $z = 0$ of domains I and II).

Usually, for the axially symmetric problems under consideration, two types of subdomains can be distinguished in $_LQ$: the interior of circular ($\rho < b$) and circular coaxial ($b < \rho < a$) semi-infinite waveguides. For them, the sets $\{\mu_n\}$, $\{\lambda_n\}$, $\{n\}$ of solutions of eigenvalue problems (3.3) are known:

$$\mu_n(\rho) = J_1(\lambda_n\rho)\sqrt{2}[bJ_0(\lambda_n b)]^{-1}, \quad n = 1, 2, \ldots,$$
$$\lambda_n > 0 \ \text{ are the roots of the equation } \ J_1(\lambda b) = 0,$$

where $\rho < b$ (TE_0 waves);

$$\mu_n(\rho) = G_1(\lambda_n, \rho)\sqrt{2}[a^2G_0^2(\lambda_n, a) - b^2G_0^2(\lambda_n, b)]^{-1/2}, \quad n = 1, 2, \ldots,$$

$$\lambda_n > 0 \text{ are the roots of the equation } G_1(\lambda, a) = 0,$$

$$G_q(\lambda, \rho) = J_q(\lambda\rho)N_1(\lambda b) - N_q(\lambda\rho)J_1(\lambda b),$$

where $b < \rho < a$ (TE_0 waves); and

$$\mu_n(\rho) = J_1(\lambda_n\rho)\sqrt{2}[bJ_1(\lambda_n b)]^{-1}, \quad n = 1, 2, \ldots,$$

$$\lambda_n > 0 \text{ are the roots of the equation } J_0(\lambda b) = 0,$$

where $\rho < b$ (TM_0 waves),

$$\mu_n(\rho) = \tilde{G}_1(\lambda_n, \rho)\sqrt{2}[a^2\tilde{G}_1^2(\lambda_n, a) - b^2\tilde{G}_1^2(\lambda_n, b)]^{-1/2}, \quad n = 1, 2, \ldots,$$

$$\mu_0(\rho) = [\rho\sqrt{\ln(a/b)}]^{-1},$$

$$\lambda_n > 0 \quad (n = 1, 2, \ldots) \text{ are the roots of the equation } \tilde{G}_0(\lambda, b) = 0, \quad \lambda_0 = 0,$$

$$\tilde{G}_q(\lambda, \rho) = J_q(\lambda\rho)N_0(\lambda a) - N_q(\lambda\rho)J_0(\lambda a),$$

where $b < \rho < a$ (TM_0 waves). Here, J_q and N_q are the Bessel and Neumann functions.

3.2. Exact Absorbing Conditions

The evolutionary bases $w(z, t) = \{w_n(z, t)\}$ of the signals that propagate through segments of closed waveguide channels and Floquet channels are qualitatively similar (see [5, 6, 9, 10]). They are described by similar initial-boundary value problems for Klein–Gordon equations (a characteristic example is given by problems (2.5) and (3.5)). This enables us to simply repeat (up to the notation and some unimportant details) nonlocal conditions (2.18)–(2.20) obtained in Section 2. All the procedures for constructing exact ABCs, namely,

$$U(\rho, 0, t) = -\sum_n \left\{ \int_0^t J_0[\lambda_n(t-\tau)]\left[\frac{\partial}{\partial z}\langle U(z, \tilde{\rho}, \tau), \mu_n(\tilde{\rho})\rangle\right]\Big|_{z=0} d\tau \right\}\mu_n(\rho) = V_1(\rho, t), \tag{3.6}$$
$$\rho_1 \le \rho \le \rho_2, \quad t \ge 0,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)U(\rho, z, t)\Big|_{z=0} = \sum_n \left\{ \int_0^t J_1[\lambda_n(t-\tau)]\left[\frac{\partial}{\partial z}\langle U(z, \tilde{\rho}, \tau), \mu_n(\tilde{\rho})\rangle\right]\Big|_{z=0} d\tau \right\}\lambda_n\mu_n(\rho) = V_2(\rho, t), \tag{3.7}$$
$$\rho_1 \le \rho \le \rho_2, \quad t \ge 0,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)U(\rho, z, t)\Big|_{z=0} = -\sum_n \left\{ \int_0^t J_1[\lambda_n(t-\tau)](t-\tau)^{-1}\langle U(0, \tilde{\rho}, \tau), \mu_n(\tilde{\rho})\rangle d\tau \right\}\lambda_n\mu_n(\rho) = V_3(\rho, t), \tag{3.8}$$
$$\rho_1 \le \rho \le \rho_2, \quad t \ge 0,$$

for energy offtake channels in axially symmetric waveguide units, are almost identical to those that were considered for the case of two-dimensional (in the plane $y0z$) model problems (2.1). In these (and the following) formulas for the circular waveguide, we must set $\rho_1 = 0$ and $\rho_2 = b$; for the coaxial waveguide, we set $\rho_1 = b$ and $\rho_2 = a$.

To pass from (3.6)–(3.8) to local conditions of type (2.25)–(2.27), we sum over all $n \in \{n\}$ the exact local ABCs for particular partial components (modes) of the outgoing nonsinusoidal waves, which are preliminary multiplied by $\mu_n(\rho)$, and then use the following representations for the auxiliary functions $u_n(t, \varphi)$ and

$W(\rho, t, \varphi)$ (and use the same procedure for the functions $w_n(z, t)$ and $U(\rho, z, t)$), which follow from (3.3) and (3.4):

$$\sum_n \lambda_n^2 u_n(t, \varphi)\mu_n(\rho) = -\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho W(\rho, t, \varphi), \quad W(\rho, t, \varphi) = \sum_n u_n(t, \varphi)\mu_n(\rho).$$

As a result, we obtain

$$U(\rho, 0, t) = \frac{2}{\pi}\int_0^{\pi/2}\frac{\partial W(\rho, t, \varphi)}{\partial t}d\varphi, \quad t \ge 0, \quad \rho_1 \le \rho \le \rho_2,$$

$$\left(\frac{\partial^2}{\partial t^2} - \sin^2\varphi\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\right)W(\rho, t, \varphi) = -\frac{\partial U(\rho, z, t)}{\partial z}\bigg|_{z=0}, \quad \rho_1 < \rho < \rho_2, \quad t > 0, \tag{3.9}$$

$$W(\rho, 0, \varphi) = \frac{\partial W(\rho, t, \varphi)}{\partial t}\bigg|_{t=0} = 0, \quad \rho_1 \le \rho \le \rho_2;$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)U(\rho, z, t)\bigg|_{z=0} = \frac{2}{\pi}\int_0^{\pi/2} W(\rho, t, \varphi)\cos^2\varphi d\varphi, \quad t \ge 0, \quad \rho_1 \le \rho \le \rho_2,$$

$$\left(\frac{\partial^2}{\partial t^2} - \cos^2\varphi\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\right)W(\rho, t, \varphi) = -\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left[\frac{\partial}{\partial z}U(\rho, z, t)\bigg|_{z=0}\right], \quad \rho_1 < \rho < \rho_2, \quad t > 0, \tag{3.10}$$

$$W(\rho, 0, \varphi) = \frac{\partial W(\rho, t, \varphi)}{\partial t}\bigg|_{t=0} = 0, \quad \rho_1 \le \rho \le \rho_2;$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)U(\rho, z, t)\bigg|_{z=0} = \frac{2}{\pi}\int_0^{\pi/2}\frac{\partial W(\rho, t, \varphi)}{\partial t}\sin^2\varphi d\varphi, \quad t \ge 0, \quad \rho_1 \le \rho \le \rho_2,$$

$$\left(\frac{\partial^2}{\partial t^2} - \cos^2\varphi\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\right)W(\rho, t, \varphi) = \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho U(\rho, 0, t), \quad \rho_1 < \rho < \rho_2, \quad t > 0, \tag{3.11}$$

$$W(\rho, 0, \varphi) = \frac{\partial W(\rho, t, \varphi)}{\partial t}\bigg|_{t=0} = 0, \quad \rho_1 \le \rho \le \rho_2.$$

The initial-boundary value problems in the auxiliary functions $W(\rho, t, \varphi)$ must be supplemented with the following boundary conditions for all observation times t ($t \ge 0$):

$$W(0, t, \varphi) = W(b, t, \varphi) = 0 \quad \text{for TE}_0 \text{ waves},$$

$$W(0, t, \varphi) = \frac{\partial(\rho W(\rho, t, \varphi))}{\partial\rho}\bigg|_{\rho=b} = 0 \quad \text{for TM}_0 \text{ waves}$$

(for domain I corresponding to a circular waveguide) and

$$W(b, t, \varphi) = W(a, t, \varphi) = 0 \quad \text{for TE}_0 \text{ waves},$$

$$\frac{\partial(\rho W(\rho, t, \varphi))}{\partial\rho}\bigg|_{\rho=b} = \frac{\partial(\rho W(\rho, t, \varphi))}{\partial\rho}\bigg|_{\rho=a} = 0 \quad \text{for TM}_0 \text{ waves}$$

(for domain II corresponding to a coaxial circular waveguide).

4. VECTOR PROBLEMS

When examining an arbitrary waveguide unit, we will not consider the bounded part of the space in which all the inhomogeneities and sources are located; in this part, the corresponding vector telegraph equation subject to usual boundary and initial conditions can be solved by the standard well-known methods (see [22, 23]). We are interested in channels (closed semi-infinite regular waveguides) through which the signals produced by the unit propagate. Consider one such channel (see Fig. 3). In the domain ${}_LQ = \{g = \{x, y, z\}$:

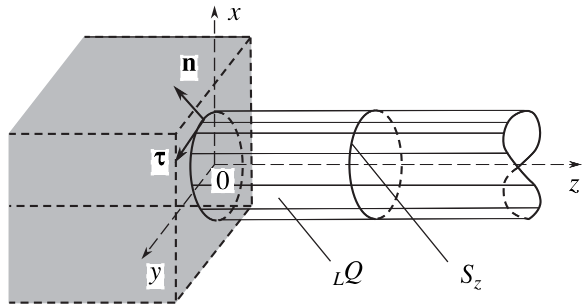

Fig. 3.

$\{x, y\} \in \operatorname{int}S_z, z > 0\}$, the field $\mathbf{E}(g, t)$, which is nonzero in this domain only for $t > 0$, is a wave propagating in the direction $z = \infty$; it satisfies the vector initial-boundary value problem

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta\right)\mathbf{E}(g, t) = 0, \quad t > 0, \quad g \in {}_LQ,$$
$$\mathbf{E}(g, t)\big|_{t=0} = \frac{\partial \mathbf{E}(g, t)}{\partial t}\bigg|_{t=0} = 0, \quad g \in {}_L\bar{Q}, \tag{4.1}$$
$$E_z(g, t)\big|_{g \in S} = (\boldsymbol{\tau} \cdot \mathbf{E}_\perp(g, t))\big|_{g \in S} = 0, \quad t \ge 0.$$

Here (see Fig. 3), $\mathbf{E} = E_z\mathbf{z} + \mathbf{E}_\perp$, $\mathbf{E}_\perp = E_x\mathbf{x} + E_y\mathbf{y}$; S_z is the boundary contour of the cross section of the regular waveguide; $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\boldsymbol{\tau}$, and $\mathbf{n}$ are the unit vectors on the coordinate axes, the tangent, and the normal to S_z; $\operatorname{int}S_z$ is the domain in the plane $z = \text{const}$ bounded by S_z; $S = S_z \times (0 < z < \infty)$; and Δ is the Laplace operator, which is written as $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ in the Cartesian coordinates.

Using the scalar Borgnis functions $U^E(g, t)$ and $U^H(g, t)$ (see [21]) such that

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta\right)\frac{\partial U^{E,H}(g, t)}{\partial t} = 0, \quad t > 0, \quad g \in {}_LQ,$$
$$U^E(g, t)\big|_{g \in S} = \frac{\partial U^H(g, t)}{\partial \mathbf{n}}\bigg|_{g \in S} = 0, \quad t \ge 0, \tag{4.2}$$

we can represent the general solution to the vector differential equation in problem (4.1) by

$$E_x = \frac{\partial^2 U^E}{\partial x \partial z} - \frac{\partial^2 U^H}{\partial y \partial t}, \quad E_y = \frac{\partial^2 U^E}{\partial y \partial z} + \frac{\partial^2 U^H}{\partial x \partial t}, \quad E_z = \frac{\partial^2 U^E}{\partial z^2} - \frac{\partial^2 U^E}{\partial t^2}. \tag{4.3}$$

Let us verify that the field $\mathbf{E}(g, t)$ thus represented also satisfies the boundary conditions of problem (4.1):

$$(\boldsymbol{\tau} \cdot \mathbf{E}_\perp) = \left(\frac{dx}{d\theta}\frac{\partial}{\partial x} + \frac{dy}{d\theta}\frac{\partial}{\partial y}\right)\frac{\partial}{\partial z}U^E - \left(\frac{dx}{d\theta}\frac{\partial}{\partial y} - \frac{dy}{d\theta}\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}U^H = \frac{d}{d\theta}\left(\frac{\partial}{\partial z}U^E\right) - \frac{\partial}{\partial \mathbf{n}}\left(\frac{\partial}{\partial t}U^H\right) = 0.$$

Here, we used the parametrization $S_z = S_z(\theta) = \{x(\theta), y(\theta)\}_\theta$ of the contour S_z and the following representation of the tangent and normal vectors to S_z (see [18]):

$$\boldsymbol{\tau} = \frac{dx}{d\theta}\mathbf{x} + \frac{dy}{d\theta}\mathbf{y}, \quad \mathbf{n} = -\frac{dy}{d\theta}\mathbf{x} + \frac{dx}{d\theta}\mathbf{y}.$$

Separate out the transverse variables x and y in problem (4.2) and write the solution in the form

$$U^{E,H}(x, y, z, t) = \sum_{n=1,0}^{\infty} w_n^{E,H}(z, t)\mu_n^{E,H}(x, y), \tag{4.4}$$

where $\{\mu_n^E(x, y)\}_{n=1}^{\infty}$ and $\{\mu_n^H(x, y)\}_{n=0}^{\infty}$ are complete orthonormal systems of solutions of the Sturm–Liouville problems for the equations $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \lambda^2)\mu = 0$ in the domain $\mathrm{int}S_z$ subject to the Dirichlet ($\mu^E(x, y)|_{\{x, y\} \in S_z} = 0$) and Neumann ($\partial\mu^H(x, y)/\partial\mathbf{n}|_{\{x, y\} \in S_z} = 0$) conditions on its boundary S_z. With these solutions, we associate the eigenvalues λ_n^E and λ_n^H. The substitution of (4.4) into (4.3) yields the following representation for the field $\mathbf{E}(g, t)$:

$$E_z = \sum_{n=1}^{\infty} v_{n,z}(z, t)\xi_{n,z}(x, y), \quad \mathbf{E}_\perp = \sum_{n=-\infty}^{\infty} v_{n,\perp}(z, t)\boldsymbol{\xi}_{n,\perp}(x, y), \quad t \ge 0, \quad z \ge 0. \tag{4.5}$$

Then, the problem of finding the scalar functions $v_{n,z}(z, t)$ and $v_{n,\perp}(z, t)$ is reduced to solving the initial-boundary value problems

$$\begin{gathered}\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - \alpha_{n,z}^2\right)v_{n,z}(z, t) = 0, \quad t > 0, \quad z > 0,\\ v_{n,z}(z, 0) = \frac{\partial}{\partial t}v_{n,z}(z, t)\Big|_{t=0} = 0, \quad z \ge 0, \quad n = 1, 2, \dots;\end{gathered} \tag{4.6}$$

$$\begin{gathered}\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - \alpha_{n,\perp}^2\right)v_{n,\perp}(z, t) = 0, \quad t > 0, \quad z > 0,\\ v_{n,\perp}(z, 0) = \frac{\partial}{\partial t}v_{n,\perp}(z, t)\Big|_{t=0} = 0, \quad z \ge 0, \quad n = 0, \pm1, \pm2, \dots .\end{gathered} \tag{4.7}$$

Here, $\alpha_{n,z} = \lambda_n^E$; $\alpha_{n,\perp} = \lambda_n^E$ for $n = 1, 2, \dots$, $\alpha_{n,\perp} = \lambda_{-n}^H$ for $n = 0, -1, -2, \dots$, $\xi_{n,z} = \mu_n^E$; $\boldsymbol{\xi}_{n,\perp} = (\partial\mu_n^E/\partial x)\mathbf{x} + (\partial\mu_n^E/\partial y)\mathbf{y}$ for $n = 1, 2, \dots$, and $\boldsymbol{\xi}_{n,\perp} = -(\partial\mu_{-n}^H/\partial y)\mathbf{x} + (\partial\mu_{-n}^H/\partial x)\mathbf{y}$ for $n = 0, -1, -2, \dots$.

The inversion formulas

$$\begin{gathered} v_{n,z}(z, t) = \int_{\mathrm{int}S_z} E_z(x, y, z, t)\xi_{n,z}(x, y)dxdy,\\ v_{n,\perp}(z, t) = (\alpha_{n,\perp})^{-2}\int_{\mathrm{int}S_z} (\mathbf{E}_\perp(x, y, z, t) \cdot \boldsymbol{\xi}_{n,\perp}(x, y))dxdy\end{gathered} \tag{4.8}$$

for (4.5) are derived on the basis of the properties of the systems of functions $\{\mu_n^{E,H}(x, y)\}$. The first of these formulas is an obvious consequence of the fact that $\{\mu_n^E(x, y)\}$ consists of orthogonal functions with the unit norm. The second one is proved by the consideration of the integrals

$$\int_{\mathrm{int}S_z}\left[\left(\frac{\partial\mu_n^E}{\partial x}\mathbf{x} + \frac{\partial\mu_n^E}{\partial y}\mathbf{y}\right)\cdot\left(\frac{\partial\mu_m^E}{\partial x}\mathbf{x} + \frac{\partial\mu_m^E}{\partial y}\mathbf{y}\right)\right]dxdy = \int_{\mathrm{int}S_z}(\mathrm{grad}\mu_n^E \cdot \mathrm{grad}\mu_m^E)dxdy$$

$$= -\int_{\mathrm{int}S_z}\mu_m^E\Delta\mu_n^E dxdy + \int_{S_z}\mu_m^E\frac{\partial\mu_n^E}{\partial\mathbf{n}}ds = \begin{cases}0, & m \ne n,\\ (\lambda_n^E)^2, & m = n;\end{cases}$$

$$\int_{\mathrm{int}S_z}\left[\left(-\frac{\partial\mu_n^H}{\partial y}\mathbf{x} + \frac{\partial\mu_n^H}{\partial x}\mathbf{y}\right)\cdot\left(-\frac{\partial\mu_m^H}{\partial y}\mathbf{x} + \frac{\partial\mu_m^H}{\partial x}\mathbf{y}\right)\right]dxdy = \int_{\mathrm{int}S_z}([\mathrm{grad}\mu_n^H \times \mathbf{z}] \cdot [\mathrm{grad}\mu_m^H \times \mathbf{z}])dxdy$$

$$= \int_{\mathrm{int}S_z}(\mathrm{grad}\mu_n^H \cdot \mathrm{grad}\mu_m^H)dxdy = \begin{cases}0, & m \ne n,\\ (\lambda_n^H)^2, & m = n;\end{cases}$$

$$\int_{\mathrm{int}S_z}\left[\left(\frac{\partial\mu_n^E}{\partial x}\mathbf{x}+\frac{\partial\mu_n^E}{\partial y}\mathbf{y}\right)\cdot\left(-\frac{\partial\mu_m^H}{\partial y}\mathbf{x}+\frac{\partial\mu_m^H}{\partial x}\mathbf{y}\right)\right]dxdy = \int_P(d\mathbf{P}\cdot\mathrm{rot}\boldsymbol{\mu}_m^H) = \int_{S_z}(d\mathbf{S}_z\cdot\boldsymbol{\mu}_m^H) = 0.$$

Here, $\boldsymbol{\mu}_m^H = \mu_m^H\mathbf{z}$; P is the surface of the function $z = \mu_n^E(x, y)$ spanned by the contour S_z, $d\mathbf{P} = [-(\partial\mu_n^E/\partial x)\mathbf{x} - (\partial\mu_n^E/\partial y)\mathbf{y} + \mathbf{z}]dxdy$ is the vector of the surface element of P, and $d\mathbf{S}_z = \boldsymbol{\tau}d\theta$ is the vector element of S_z.

Problems (4.6) and (4.7) are identical to problems (2.5) and (3.5), which were thoroughly considered above. Therefore, we drop the details and proceed to the analysis of their solutions:

$$\nu_n(l, t) = -\int_0^t J_0[\alpha_n(t-\tau)]\nu'_n(l,\tau)d\tau, \quad l \ge 0, \quad t \ge 0, \tag{4.9}$$

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right)\nu_n(z, t)\bigg|_{z=l} = \alpha_n\int_0^t J_1[\alpha_n(t-\tau)]\nu'_n(l,\tau)d\tau, \quad l \ge 0, \quad t \ge 0, \tag{4.10}$$

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right)\nu_n(z, t)\bigg|_{z=l} = -\alpha_n\int_0^t J_1[\alpha_n(t-\tau)](t-\tau)^{-1}\nu_n(l,\tau)d\tau, \quad l \ge 0, \quad t \ge 0. \tag{4.11}$$

If we set $\alpha_n = \alpha_{n,z}$ ($n = 1, 2, \ldots$) in (4.9)–(4.11), then we arrive at the exact conditions for the elements $\nu_{n,z}(z, t)$ of the evolutionary basis of the component $E_z(g, t)$ of the outgoing wave $\mathbf{E}(g, t)$. For $\alpha_n = \alpha_{n,\perp}$ ($n = 0, \pm1, \pm2, \ldots$), we obtain similar representations for the elements $\nu_{n,\perp}(z, t)$ (the amplitudes of the partial components of the transverse electric field $\mathbf{E}_\perp(g, t)$ of the wave $\mathbf{E}(g, t)$). Below, we drop the subscripts z and $\perp$ in the eigenvalues α_n and the functions ν_n, ξ_n, and E, as well as the difference between vector and scalar functions. Indeed, the results following from (4.9)–(4.11) are unambiguously interpreted both for the longitudinal field $E_z(g, t)$ and the transverse field $\mathbf{E}_\perp(g, t)$.

Conditions (4.9)–(4.11) are nonlocal in terms of time. The use of these conditions in the finite difference method for solving the general problem of finding the field $E(g, t)$ in the waveguide unit induces a space nonlocality: at every time step, it is necessary to pass from series (4.5) to amplitudes (4.8) and back. Using the reasoning in Sections 2 and 3, we replace conditions (4.10) and (4.11) by the following conditions that are local in space and time:

$$\begin{gathered}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)E(g, t)\bigg|_{z=l} = \frac{1}{\pi}\int_0^\pi W(x, y, t, \varphi)d\varphi, \quad \{x, y\}\in\overline{\mathrm{int}S_z}, \quad t \ge 0,\\
\left[\frac{\partial^2}{\partial t^2}-\sin^2\varphi\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]W(x, y, t, \varphi) = -\sin^2\varphi\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left[\frac{\partial E(g,t)}{\partial z}\bigg|_{z=l}\right],\\
\{x, y\}\in\mathrm{int}S_z, \quad t > 0,\\
W(x, y, 0, \varphi) = \frac{\partial W(x, y, t, \varphi)}{\partial t}\bigg|_{t=0} = 0, \quad \{x, y\}\in\overline{\mathrm{int}S_z};\end{gathered} \tag{4.12}$$

$$\begin{gathered}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)E(g, t)\bigg|_{z=l} = \frac{2}{\pi}\int_0^{\pi/2}\frac{\partial W(x, y, t, \varphi)}{\partial t}\sin^2\varphi d\varphi, \quad \{x, y\}\in\overline{\mathrm{int}S_z}, \quad t \ge 0,\\
\left[\frac{\partial^2}{\partial t^2}-\cos^2\varphi\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]W(x, y, t, \varphi) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)E(x, y, l, t),\\
\{x, y\}\in\mathrm{int}S_z, \quad t > 0,\\
W(x, y, 0, \varphi) = \frac{\partial W(x, y, t, \varphi)}{\partial t}\bigg|_{t=0} = 0, \quad \{x, y\}\in\overline{\mathrm{int}S_z}.\end{gathered} \tag{4.13}$$

We do not write out the condition following from (4.9), which is similar to condition (2.25). The auxiliary scalar or vector functions $W(x, y, t, \varphi)$ in (4.12) and (4.13) are expressed in terms of the same bases of transverse functions $\xi_n(x, y)$ as the field $E(g, t)$:

$$W(x, y, t, \varphi) = \sum_n u_n(t, \varphi)\xi_n(x, y). \tag{4.14}$$

Formally, the auxiliary initial-boundary value problems in (4.12), (4.13) must be complemented with boundary conditions (on the contour S_z) for the functions $W(x, y, t, \varphi)$ and $\mathbf{W}(x, y, t, \varphi)$ $(t \ge 0)$:

$$W(x, y, t, \varphi)|_{\{x, y\} \in S_z} = 0, \quad E(g, t) \longrightarrow E_z(g, t),$$

$$(\boldsymbol{\tau} \cdot \mathbf{W}(x, y, t, \varphi))|_{\{x, y\} \in S_z} = 0, \quad E(g, t) \longrightarrow \mathbf{E}_\perp(g, t).$$

These conditions are an obvious consequence of representations (4.14). The exact absorbing conditions given above for case (4.13 are obtained using the same technique as that used for deriving ABCs (2.27); when proceeding from (4.10) to (4.12), we modified the intermediate representations that were earlier used to obtain ABCs (2.26): here,

$$J_1(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\varphi)\sin\varphi d\varphi \text{ (see [19])}$$

and

$$u_n(t, \varphi) = \alpha_n \sin\varphi \int_0^t \sin[\alpha_n(t-\tau)\sin\varphi]\nu'_n(l, \tau)d\tau, \quad t \ge 0, \quad 0 \le \varphi \le \pi.$$

Accordingly, the result (cf. (2.26) and (4.12)) also changed.

5. CONCLUSIONS

In this paper, we considered the main technical details of the approach reducing open two-dimensional scalar and three-dimensional vector initial-boundary problems concerning the transformation of nonsinusoidal waves by compact waveguide units to equivalent closed problems. The procedure of constructing the exact absorbing conditions for the coordinate virtual boundaries in the cross sections of semiinfinite offtaking waveguides is based on a sequence of techniques that are widely used in the theory of hyperbolic equations [21]: incomplete separation of variables in problems for telegraph equations $\longrightarrow$ integral transforms in problems for the one-dimensional Klein–Gordon–Fock equation $\longrightarrow$ the solution of auxiliary boundary value problems for ordinary differential equations $\longrightarrow$ inverse integral transforms.

Exact ABCs make it possible to correctly restrict the domain of computations (the discretization domain of the given initial-boundary value problems) to Q_L. The implementation of the corresponding algorithm yields simple and accurate numerical solutions for all observation times t as in the case of physically closed domains Q_L.

Not all the capabilities of the approach proposed in this paper were considered here. Insignificant variations in technical details and the sequence of basic operations can yield new results as compared to those presented in this paper. The development of new absorbing conditions is constantly stimulated by computational practice and new problems.

We want to stress the following important differences in the nonlocal and local exact absorbing conditions. The nonlocal conditions require (see, e.g., Section 4) that complete information about the systems of eigenfunctions and eigenvalues of the Sturm–Liouville operator in the domain intS_z with the Dirichlet and Neumann conditions on the boundary S_z be available. The solution of the corresponding eigenvalue problems for arbitrary S_z can be very tedious and require considerable computer resources. The local conditions are free of this drawback. They are preferable when dealing with units with energy offtaking channels in which the transverse functions cannot be found analytically. But even in the case when the transverse functions of regular waveguide channels are known, the use of local conditions makes the computations much faster and requires less computer memory.

We confirm the reasoning above using a simple example. The implementation of a finite difference analog (the three-layer explicit scheme with the second order of approximation) for initial-boundary problem

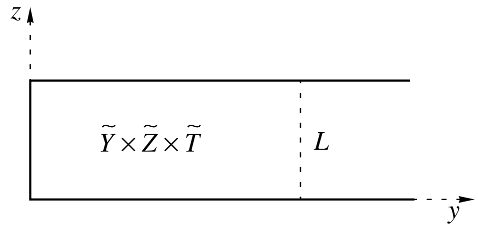

Fig. 4.

(2.1) subject to nonlocal condition (2.18) for the simplest waveguide resonator depicted in Fig. 4 requires

$$\tilde{N} = [(5\tilde{X}\tilde{Y}\tilde{T})_+ + (6\tilde{X}\tilde{Y}\tilde{T})_\times]_1 + [(\tilde{M}[0.5\tilde{T}^2 + \tilde{T}(0.5 + 2\tilde{Y})])_+ + (\tilde{M}[\tilde{T}^2 + \tilde{T}(1 + 3\tilde{Y})])_\times]_2 \tag{5.1}$$

addition (the subscript +) and multiplication (the subscript ×) operations and

$$\tilde{P} = [2\tilde{X}\tilde{Y}]_1 + [\tilde{M}(2\tilde{T} + \tilde{Y})]_2 \tag{5.2}$$

operations of memory allocation. Here, the computational efforts are divided into groups corresponding to the standard scheme in the domain Q_L with the physical boundaries S (the subscript 1) and to the exact absorbing conditions on the virtual boundary L (the subscript 2). The variables $\tilde{Y}$, $\tilde{Z}$, and $\tilde{T}$ determine the size of the computational space in terms of the number of grid points on the axes y, z and t; $\tilde{M}$ is the number of harmonics that are taken into account in representations (2.2) (in the case of nonlocal conditions) or the number of values of φ on L (in the case of local conditions).

In the case of local conditions (2.25), we have

$$\tilde{N} = [(5\tilde{X}\tilde{Y}\tilde{T})_+ + (6\tilde{X}\tilde{Y}\tilde{T})_\times]_1 + [(\tilde{T}\tilde{Y}[1 + 6\tilde{M}])_+ + (\tilde{T}\tilde{Y}[1 + 7\tilde{M}])_\times]_2, \tag{5.3}$$

$$\tilde{P} = [2\tilde{X}\tilde{Y}]_1 + [2\tilde{Y}(\tilde{M} + 1)]_2. \tag{5.4}$$

Comparing (5.1), (5.2) with (5.3), (5.4), we see that the use of local conditions yields a considerable savings in the computational cost.

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