# Lasing Spectra and Thresholds of the Whispering-Gallery Modes in a Circular Dielectric Microcavity

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Microlaser designs based on high-reflectivity whispering-gallery (WG) modes around the edge of a thin semiconductor microdisk have been studied since the early 1990's [1-3]. Optical pump is normally arranged with a wide external laser beam [1,2], hence the gain over the disk can be considered as uniform. The same, although less justified, holds for injection lasers [3]. Mathematically, these modes are the source-free solutions of the 3-D Maxwell equations, however if the disk thickness is only a fraction of the wavelength the modes can be studied in 2-D formulation. Still surprisingly, it appears that accurate study of the circular-cavity lasing modes is absent, as the simplified analysis of [2] was based on rough assumptions, and FDTD simulations of [3] did not address the modes directly.

In this paper we study the lasing eigenvalue problem (LEP) for a circular resonator [4]. We look for the non-attenuating time-harmonic electromagnetic field $\sim e^{-ikct}$, $k=\mathrm{Re}k>0$ in and out of a dielectric circular cylinder of radius $a$. We assume that the field does not vary along the $z$ axis and can be characterized by a scalar function $U$, which represents either $E_z$ or $H_z$ component depending on the polarization. Off the boundaries, this function must satisfy the Helmholtz equation $\left[\Delta + k^2\nu^2(r,\phi)\right]U(r,\phi)=0$. Step-wise function $\nu(r,\phi)$ is assumed 1 outside the cavity and a complex value inside: $\nu=\alpha-i\gamma$, where $\alpha>0$ is the refraction index and $\gamma>0$ is the material gain. The field must satisfy the continuity conditions across the boundary of resonator. In view of the real value of the wavenumber $k$, we impose the Sommerfeld radiation condition at infinity ($r$ ). The eigenvalues are considered as pairs of parameters $(\kappa,\gamma)$. The first of them is the normalized frequency of lasing, $\kappa=ka$, while the second is the threshold gain. This formulation is different from the "classical" formulation of eigenvalue problem for an open cavity, when the complex-valued frequency $k$ is eigenvalue parameter [5],[6]. Then, the long-living natural oscillations with high $Q$-factors (i.e., small $\mathrm{Im}k<0$) are of the main interest; however the condition at infinity should be modified to permit the field growing up. In the case of our formulation of LEP, there is no need of such admission of non-physical behavior [4]. Besides, the threshold gain directly characterizes a laser operation while the $Q$-factor makes this indirectly.

For a circular resonator, separation of variables splits eigenvalues into families according to the azimuth index $n$. This reduces LEP to the set of independent equations in terms of the real and complex-argument cylindrical functions of integer index, $n$=0,1,2, ...

$$J_n'(ka\nu)H_n^{(1)}(ka)-\beta H_n'^{(1)}(ka)J_n(ka\nu)=0, \text{ where } \beta=\begin{cases}\nu, U=E_z\\ \nu^{-1}, U=H_z\end{cases}$$

The theory of complex variables tells that the set of eigenvalues $(\kappa_{nm},\gamma_{nm}), n=0,1,\dots, m=1,2,\dots$ is discrete; each of them may have only finite multiplicity; there are no finite accumulation points of eigenvalues. All $\gamma_{nm}>0$ [4]. Further we use 2-D Newton's method to obtain the eigenvalues numerically. In computations, we assume that the refraction index is $\alpha=3.53$ that corresponds to GaAs.

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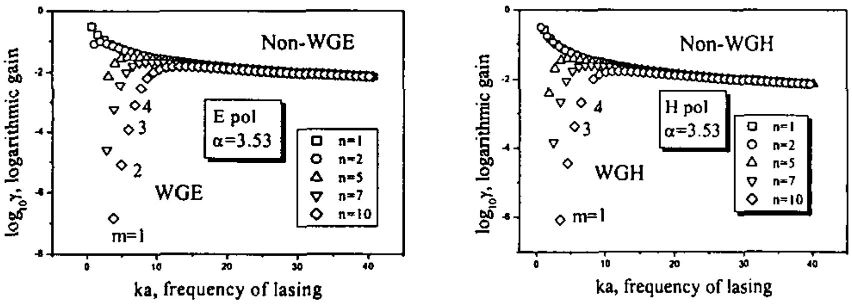



*Fig. 1. The eigenvalue pairs for a GaAs circular cavity.*

Fig. 1 shows the eigenvalue pairs in the plane $(\kappa, \gamma)$ within the strip $0 < \kappa = ka < 40$. Each $n$-th family of modes displays two different types of behavior depending on the lasing frequency. If $n/\alpha < ka < n$, then the modes are WG ones and have exponentially small thresholds. This is explained by the quasi-total-reflection mechanism of the WG mode field forming. It is seen that the higher the azimuth index $n$ of the lasing mode, the smaller the threshold gain. The smallest threshold in each family is observed for the $WG_{n1}$ mode, whose E-field has a single maximum inside or near the cavity. The type of modal behavior changes if $ka$ approaches $n$. Much larger values of $\gamma_{nm}$ are observed if $ka > n$, therefore we call corresponding modes as non-WG ones; in this range, thresholds are inverse proportional to the lasing frequencies. Interestingly, it appears that for very small cavities, namely if $ka < n/\alpha$, no lasing modes of the $n$-th family can be found.

Each eigenvalue continuously depends on refraction index $\alpha$. As one can see in Fig. 2, the threshold gains and lasing frequencies of the WG modes get smaller with greater values of $\alpha$.

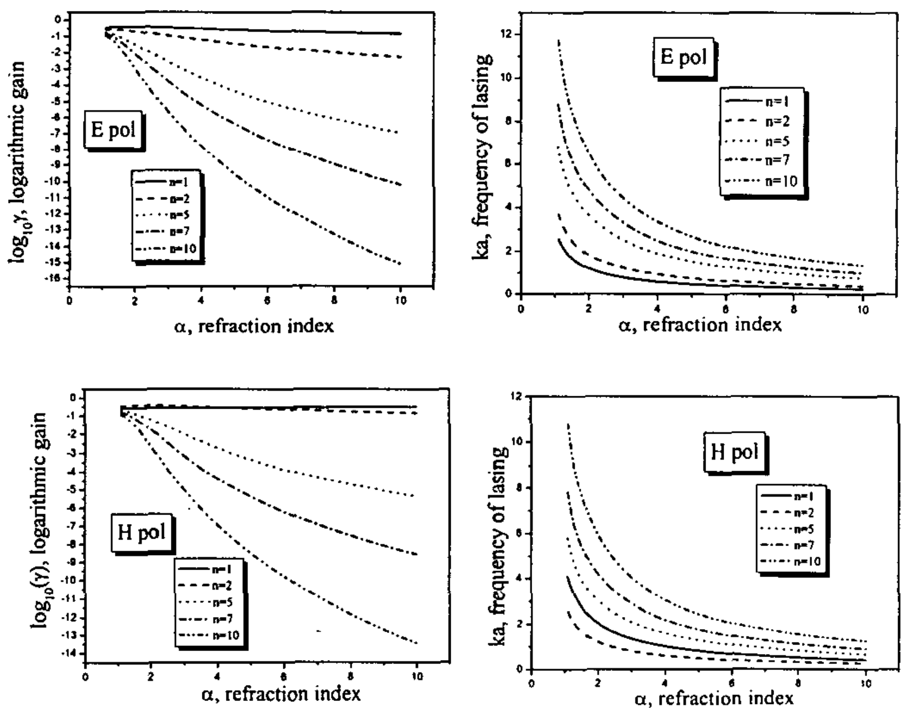



*Fig. 2. Dependences of the characteristics of the $WGE_{nl}$ and $WGH_{nl}$ modes on the refraction index $\alpha$.*

By using the asymptotic expressions for cylindrical functions, we have built some analytical formulas showing clearly the mentioned above features of the non-WG and WG lasing modes, respectively:

$$\begin{cases} \kappa_{np} \approx \pi(p+n+0.5)/2\alpha \\ \gamma_{np} \approx \dfrac{1}{2\kappa_{np}} \ln \dfrac{\alpha-1}{\alpha+1} \end{cases}, \kappa > n \quad \text{and} \quad \begin{cases} \kappa_{np} \approx \pi(p+n+0.5)/2\alpha \\ \gamma_{np} \approx \dfrac{\alpha}{n-1}\left(\dfrac{e\kappa_{np}}{2n}\right)^{2n} \end{cases}, n/\alpha < \kappa < n$$

where $p=2j$ for the E-polarization or $p=2j+1$ for the H-polarization, $j$ is integer.

The modal field patterns of the circular microcavity are given by the following functions:

$$U_{nm}(r,\varphi) = \begin{cases} \dfrac{H_n^{(1)}(\kappa_{nm})}{J_n(\kappa_{nm}\nu_{nm})} J_n(\kappa_{nm}\nu_{nm}\rho)\cos n\varphi, \rho < 1 \\ H_n^{(1)}(\kappa_{nm}\rho)\cos n\varphi, \rho = r/a > 1, \kappa_{nm} = k_{nm}a \end{cases}.$$

Fig. 3 shows near-field intensities of the non-WG and nearly WG modes of the both polarizations.

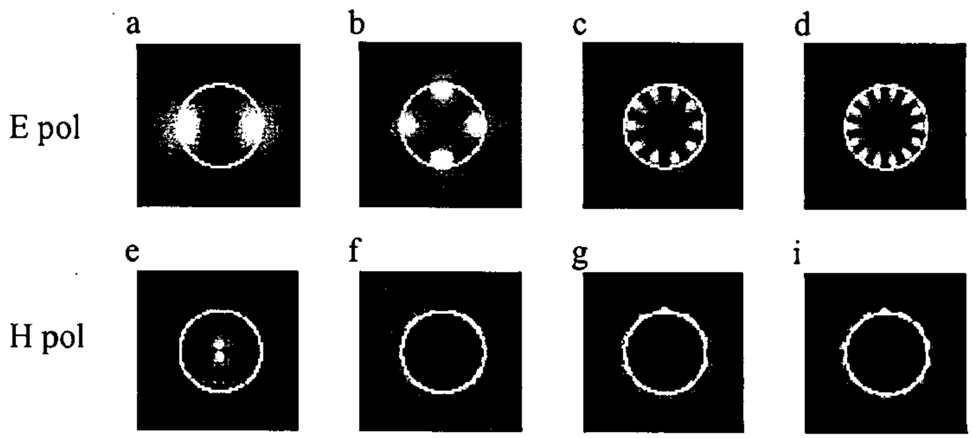



*Fig. 3. Near E-field patterns (a) $E_{1,1}$, ka=0.66, γ=0.3, (b) $E_{2,1}$, ka=1.04, γ=7.93*10$^{-2}$,* (c) $E_{5,1}$, $ka$=2.12, $\gamma$=7.7*10$^{-4}$, (d) $E_{7,1}$, $ka$=2.79, $\gamma$=2.67*10$^{-5}$, (e) $H_{1,1}$, $ka$=1.13, $\gamma$=0.27, (f) $H_{2,1}$, $ka$=0.66, $\gamma$=0.31, (g) $H_{5,1}$, $ka$=1.78, $\gamma$=3.89*10$^{-3}$, (i) $H_{7,1}$, $ka$=2.46, $\gamma$=1.46*10$^{-4}$.

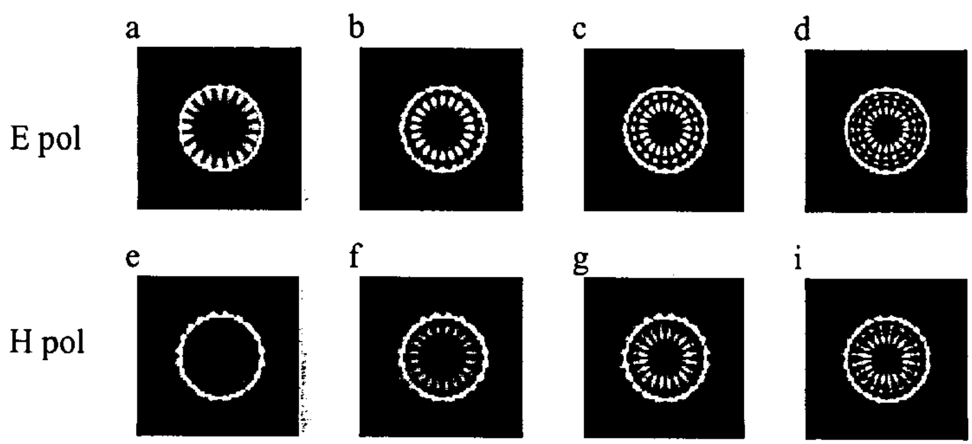
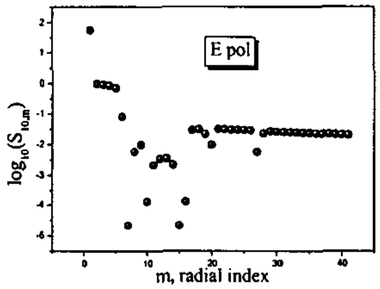



*Fig.4. Near E-field patterns of the families n=10. (a) $WGE_{10,1}$, ka=3.76, γ=1.46*10$^{-7}$, (b) $WGE_{10,2}$, ka=4.86,* $\gamma$=8.19*10$^{-6}$, (c) $WGE_{10,3}$, $ka$=5.86, $\gamma$=1.22*10$^{-4}$, (d) $WGE_{10,4}$, $ka$=6.82, $\gamma$=8.15*10$^{-4}$, (f) $WGH_{10,1}$, $ka$=3.44, $\gamma$=8.4*10$^{-7}$, (e) $WGH_{10,2}$, $ka$=4.52, $\gamma$=3.6*10$^{-5}$, (g) $WGH_{10,3}$, $ka$=5.5, $\gamma$=4.15*10$^{-4}$, (i) $WGH_{10,4}$, $ka$=6.45, $\gamma$=2.13*10$^{-3}$.

Fig. 4 shows the WG modal field patterns for the mode family $n$=10. As one can see, the modes with the larger thresholds have the fields demonstrating additional maxima along the radius.



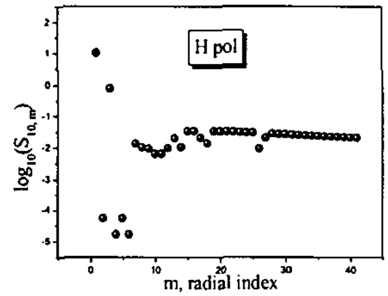




*Fig.5. The cavity stability against the mode switching.*

The cavity stability against mode switching is determined by the relative threshold difference $s_{nm} = (\gamma_{nm} - \gamma_{n'm'})/\gamma_{nm}$ among the nearest modes. Fig. 5 shows stabilities for the mode families $n$=10.

The modes with $m$=1 are the most stable in either polarization while the less stable ones have $\kappa_{nm} \approx n$.

## CONCLUSIONS

We have studied the LEP for the E and H-polarized modes in a circular dielectric microcavity. The analysis has revealed the following facts: the WG mode fields have the quasi-total-reflection mechanism and exponentially small thresholds getting down with index $n$. Non-WG modes have the frequencies and thresholds coupled by a hyperbolic relation; their thresholds are much higher than for WG modes. The ranges of the WG and non-WG mode spectra are divided by the value $\kappa \approx n$, thus $n_0\alpha^{-1} < 2\pi a\lambda_{nm}^{-1} < n$ for the WG modes and $n < 2\pi a\lambda_{nm}^{-1}$ for the non-WG ones. In each $n$-th family of either polarization, $WG_{n1}$ ($m$=1) mode is the most stable and has the lowest threshold. In the range $\kappa < n/\alpha$, lasing modes are not found. Finally, we have observed that the material-gain thresholds of the E-polarized WG modes are considerably (by an order) smaller that those of the H-polarized modes of similar field pattern.

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