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MATHEMATICAL AND NUMERICAL ANALYSIS OF THE GENERALIZED COMPLEX-FREQUENCY EIGENVALUE PROBLEM FOR TWO-DIMENSIONAL OPTICAL MICROCAVITIES*

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Abstract. The current paper proposes a parametric eigenvalue problem for the Helmholtz equation on the plane based on two well-known physical models of emission from two-dimensional (2-D) microcavity lasers. The first model has been developed for passive, either lossy or lossless, open cavities and is usually referred to as the Complex-Frequency Eigenvalue Problem. The second model, named the Lasing Eigenvalue Problem (LEP), has been tailored to characterize emission from open cavities, filled in with gain material, on the threshold of nonattenuating in time radiation. Our generalized model contains both of them as special cases. We reduce the original problem to a nonlinear eigenvalue problem for a set of boundary integral equations with weakly singular kernels and formulate it as a parametric eigenvalue problem for a holomorphic Fredholm operator-valued function, which is convenient for mathematical and numerical analysis. Using this formulation and fundamental results of the theory of holomorphic operator-valued functions, we study the properties of the spectrum. For numerical solution of the problem, we propose a Nyström method, prove its convergence, and derive error estimates in the eigenvalue approximation. Combining this discretization with the residual inverse iteration technique, we compute approximate solutions of LEP and compare them with the exact ones and with the results obtained using other numerical methods.

Key words. microcavity laser, nonlinear eigenvalue problem, Müller boundary integral equations, Nyström method

AMS subject classifications. 65N25, 65N35, 65N38, 65N80

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1. Introduction.

Microcavity lasers have been studied experimentally and theoretically since the 1990s (see, e.g., [5], [14], [25], [43]). Most of the authors used a physical model based on the search of complex-valued natural frequencies of passive open (lossless and lossy) cavities—this is usually called the Complex-Frequency Eigenvalue Problem (CFEP). As is easy to see, traditional modal analysis of a passive cavity neglects the presence of gain material and delivers mode frequencies and Q-factors only. Hence it is unable to determine and compare the most important characteristic of laser mode—the threshold gain. To overcome this significant drawback, several approaches were proposed (see review in [28]). One of them is the analysis of on-threshold modes of microcavity lasers with the aid of a modified electromagnetic eigenvalue problem, specifically tailored to extract the threshold values of gain in addition to the emission frequencies, as components of the two-component eigenvalues. Such a modified formulation, called the Lasing Eigenvalue Problem (LEP), was first introduced in 2004 in [29] and has since gained recognition in the photonics

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community.

If a microcavity, either passive or active, is shaped as a thinner-than-wavelength flat configuration, then one can reduce the dimensionality of analysis from three-dimensional (3-D) to two-dimensional (2-D), in the median plane, using the replacement of the refractive index with its effective value [10], [25]. This is an empirical transition; however, it is in good agreement with the experimentally found fact that the emission from thin cavities is observed mostly in the cavity plane, its directionality being controlled by the shape of the contour. Thin flat microcavities may have various shapes, and the above mentioned review papers provide references to the numerous efforts of researchers to find, using CFEP as a framework, a shape of the 2-D contour that improves the directionality of emission. Still, it is clear that such improvement must not be achieved at the expense of dramatic growth in threshold, which is not accessible in CFEP. Therefore, LEP is not only a mathematical model, which is more adequate to a laser, although only on the threshold, but is also an attractive instrument in the numerical optimization accounting both for directivity and threshold of lasing [31], [34] [45], [46].

Significant progress has been achieved for 2-D microcavities in [31], where LEP was reduced to a nonlinear spectral problem for the set of boundary integral equations (BIEs) named after Müller (see Chap. VI. of [23]). This set is a reliable and efficient tool for the analysis of electromagnetic modes both of the laser cavities with active regions, within LEP, and of the passive open resonators, within CFEP [4]. In [31], the Müller BIEs were solved accurately by the Nyström method, and the numerical experiments demonstrated the exponential convergence of the approximate eigenvalues to exact ones, with progressively larger discretization orders. A similar study was performed in [39] for traditional CFEP; however, with the aid of a less economic discretization technique.

The authors of the above papers concentrated their research on the features of the proposed algorithms and physical interpretation of numerical results rather than on the mathematical aspects including the properties of the spectra of eigenvalues and the convergence analysis. The main idea of the present article is to conduct a thorough mathematical study of CFEP and LEP as well as to provide a rigorous proof of convergence of the Nyström method. Our consideration is based on the fundamental results of the theory of holomorphic operator-valued functions in a pair of Banach spaces (see, e.g., Appendix in [20] and references therein). The concept of eigenvalue problems for the holomorphic Fredholm operator-valued functions supplies an important tool for the numerical analysis of approximations of such eigenvalue problems. This analysis has a long tradition [13], [16], [17], [40], [42]. It has been used for proving convergence of various numerical algorithms for nonlinear spectral problems in many applications [8], [9], [11], [15], [32], [37]. We apply it for the Nyström method within CFEP and LEP. Thus, the results of our paper extend significantly the range of microdevices that allow a study by fully grounded mathematical methods.

First of all, we propose a more general parametric eigenvalue problem for the Helmholtz equation on the plane than CFEP and LEP. It contains both of them as special cases. We call it the generalized CFEP (GCFEP). Its complex eigenvalues k are possible values of the free-space wavenumber. They depend on the real-valued loss/gain index $\gamma \in \mathbb{R}$, i.e., the imaginary part of the refractive index. If $\gamma = 0$, then the cavity is lossless. Negative values of γ correspond to losses. If $\gamma \leq 0$, then the statement of GCFEP corresponds exactly to CFEP for a passive cavity. If for a positive γ there exists a positive eigenvalue k, then this γ is the threshold value of gain, and the pair (k, γ) together with the corresponding eigenfunction satisfy all the conditions of LEP.

Then we use the Müller BIEs and reduce GCFEP to a parametric eigenvalue problem for a holomorphic Fredholm operator-valued function, which involves weakly singular integral operators. Based on this statement of the problem we investigate the qualitative properties of the spectrum: the localization on the corresponding Riemann surface, the discreteness, the algebraic multiplicity, and the dependence of the eigenvalues k on the loss/gain parameter $\gamma \in \mathbb{R}$ (Theorem 4.4). Currently, similar spectral properties of the solutions of LEP are known only for the microcavities of the circular shape [29]. They are obtained by the method of separation of variables and the use of the theorems of complex calculus.

After that, using the Nyström method, we build a sequence of finite-dimensional holomorphic operator-valued functions that regularly approximates the original holomorphic Fredholm operator-valued function on the Riemann surface. This enables us to apply the results of [16], [17] to the numerical analysis of the proposed method (Theorems 5.1 and 5.2). We solve the obtained finite-dimensional nonlinear algebraic eigenvalue problem using a variant of the residual inverse iteration method [24] described in [33]. Analysis of numerical experiments presented in the last section of the article demonstrates exponential convergence of the Nyström method. The approximate solutions coincide with known exact solutions of LEP for circular microcavities [29] and are in good agreement with results obtained for CFEP for square [44] and triangle [12] microcavities using the finite-difference time-domain (FDTD) method.

2. Generalized complex-frequency eigenvalue problem. In this section, we formulate GCFEP for 2-D dielectric microcavities of arbitrary shape shown in Figure 1. The problem statement is based on two physical models of emission from 2-D microcavity lasers: CFEP and LEP (see, e.g., [4] and [31], respectively). We assume that the electric field $E = (E_1, E_2, E_3)$ and the magnetic field $H = (H_1, H_2, H_3)$ depend on time as $\sim exp(-ikct)$, where k is the complex-valued wavenumber and c stands for the free-space light velocity. We also assume that the fields do not vary along the x_3 axis and can be characterized by means of a scalar function u, which is the E_3 or H_3 component depending on the polarization. Suppose that all considered materials are nonmagnetic and characterized with the corresponding relative dielectric permittivity ε, or, equivalently, refractive index, $\nu = \sqrt{\varepsilon}$. Denote the complex refractive index in the bounded domain Ω_i as $\nu_i = \alpha_i - i\gamma$. Here, $\alpha_i > 0$ is the known real part of ν_i and $\gamma \in \mathbb{R}$ is the loss/gain index (a real-valued parameter of the problem). The refractive index in the unbounded domain $\Omega_e = \mathbb{R}^2 \backslash \bar{\Omega}_i$ is known and positive, $\nu_e = \alpha_e > 0$.

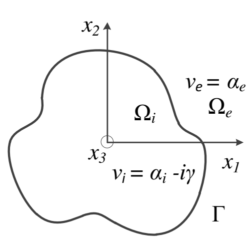

Fig. 1. *Geometry of a uniformly active* 2*-D dielectric resonator of arbitrary shape.*

Denote by R_0 the minimum value of the radius of a circle Ω_{R_0} centered at the origin such that $\Omega_i \subset \Omega_{R_0}$. We assume that the boundary Γ of the cavity is a twice continuously differentiable curve and we assume and introduce the outer normal unit vector n to that boundary. Denote by U the space of all complex-valued functions continuous on $\bar{\Omega}_i$ and $\bar{\Omega}_e$ and twice continuously differentiable on Ω_i and Ω_e. Let $\mathbb{L}$ be the Riemann surface of the function $\ln k$. For any given value of the parameter $\gamma \in \mathbb{R}$, a nonzero function $u \in U$ is referred to as an eigenfunction of GCFEP corresponding to an eigenvalue $k \in \mathbb{L}$ if the following relations are satisfied: the Helmholtz equation,

$$\Delta u(x) + k_j^2 u(x) = 0, \quad x \in \Omega_j, \quad j = i, e, \tag{2.1}$$

the transmission conditions,

$$u^- = u^+, \quad \eta_i \frac{\partial u^-}{\partial n} = \eta_e \frac{\partial u^+}{\partial n}, \quad x \in \Gamma, \tag{2.2}$$

and the outgoing Reichardt radiation condition [18], [26],

$$u(r, \varphi) = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(k_e r) \exp(il\varphi), \quad r \geq R_0. \tag{2.3}$$

Here, $k_j = k\nu_j$, $u = H_3$, $\eta_j = \nu_j^{-2}$ in the H-polarization case and $u = E_3$, $\eta_j = 1$ for the E-polarization, $j = i,\ e$; as usual, $H_l^{(1)}(z)$ is the Hankel function of the first kind and index l; r and φ are the polar coordinates of the point x; and u^- (u^+) is the limit value of the function u from inside (outside) of the boundary Γ.

Following [7, p. 68], we assume that the limit values of the normal derivative on the boundary exist in the sense that the limits

$$\frac{\partial u^\pm}{\partial n}(x) = \lim_{h \to +0} \left(n(x),\ \mathrm{grad}\ u(x \pm hn(x))\right), \quad x \in \Gamma, \tag{2.4}$$

exist uniformly on Γ. Note that for any solution u of (2.1) in Ω_e the series in (2.3) converges uniformly and absolutely on any closed domain $a \leq r \leq b$, where a and b are arbitrary numbers such that $R_0 < a < b < \infty$; and this series is infinitely termwise differentiable (see, e.g., [18]).

Within GCFEP, we are looking for the eigenvalues k on the Riemann surface $\mathbb{L}$, since the Hankel functions $H_l^{(1)}(k\nu_e r)$ are multivalued functions of the variable k, but we want to consider these functions as holomorphic (see, e.g., [18]). Denote by $\mathbb{L}_0$ the principal sheet of the Riemann surface $\mathbb{L}$ with the branch cut along the negative imaginary axis. If $k \in \mathbb{L}_0$ and $\mathrm{Im}\, k > 0$ (< 0), then u exponentially decays (grows) at infinity [18], i.e., at $r \to \infty$. If $\mathrm{Im}\, k = 0$, then the Reichardt radiation condition (2.3) is equivalent [18] to the Sommerfeld radiation condition:

$$\left(\frac{\partial}{\partial r} - ik_e\right) u = o\left(\frac{1}{\sqrt{r}}\right), \quad r \to \infty. \tag{2.5}$$

The eigenvalues k of GCFEP depend on the loss/gain index $\gamma \in \mathbb{R}$. If γ is equal to or less than zero, then the cavity is passive (lossless or lossy, respectively). For $\gamma \leq 0$, the statement of GCFEP coincides exactly with the statement of CFEP [4]. In this case, as known from the Poynting theorem (see, e.g., [4]), all the eigenvalues k are located strictly on the lower half of $\mathbb{L}_0$, i.e., have $\mathrm{Im}\, k < 0$. For positive γ, that corresponds to the gain in the cavity (active cavity), some of the eigenvalues k can belong to the real axis and the upper half of $\mathbb{L}_0$. Note also that for the time-harmonic electromagnetic field, if a complex k is an eigenvalue of GCFEP, then $-\bar{k}$

is also the eigenvalue (as usual, by $\bar{k}$ we denote the complex conjugate of k). If for some $\gamma > 0$ there exists an eigenvalue $k > 0$, then the pair (k, γ) and the corresponding eigenfunction u satisfy all the conditions of LEP [31]. Such a value of γ is the threshold value of the gain index of the cavity material that is needed to compensate for the radiation losses and provide non-attenuating in the time field function. It is important to note that these values of γ are different for different k.

THEOREM 2.1. *For each $\gamma \in \mathbb{R}$ the positive imaginary semiaxis $\mathbb{I}_+$ of the principal sheet $\mathbb{L}_0$ of $\mathbb{L}$ is free of the eigenvalues k of problem* (2.1)–(2.3).

Proof. We prove the theorem only for the H-polarized field. The proof for the E-polarization is analogous. Suppose that u is an eigenfunction of problem (2.1)–(2.3) corresponding to the eigenvalue $k = i\sigma$, where $\sigma > 0$. We apply the first Green's theorem (see, e.g., [7, p. 68]) to the functions u and $\bar{u}$ on the domains Ω_i and $\Omega_R \setminus \overline{\Omega}_i$, $R \geq R_0$. Let Ω_R be the open circle of a radius $R \geq R_0$ centered at the origin. We obtain the following equalities, respectively:

$$\int_{\Omega_i} \nabla u \cdot \nabla \bar{u} dx + \int_{\Omega_i} \bar{u} \Delta u dx = \int_{\Gamma} \bar{u}^- \frac{\partial u^-}{\partial n} dl, \tag{2.6}$$

$$\int_{\Omega_R \setminus \overline{\Omega}_i} \nabla u \cdot \nabla \bar{u} dx + \int_{\Omega_R \setminus \overline{\Omega}_i} \bar{u} \Delta u dx = -\int_{\Gamma} \bar{u}^+ \frac{\partial u^+}{\partial n} dl + \int_{\Gamma_R} \bar{u} \frac{\partial u}{\partial r} dl, \tag{2.7}$$

where "$\cdot$" is the standard inner product on $\mathbb{R}^2$ and Γ_R is the boundary of Ω_R. Taking the limit as $R \to \infty$ on the left-hand side and on the right-hand side of (2.7), we obtain

$$\int_{\Omega_e} \nabla u \cdot \nabla \bar{u} dx + \int_{\Omega_e} \bar{u} \Delta u dx = -\int_{\Gamma} \bar{u}^+ \frac{\partial u^+}{\partial n} dl. \tag{2.8}$$

Multiplying both sides of (2.6) and (2.8) with η_i and η_e, respectively, adding the resulting equations term by term, and taking into account the transmission conditions (2.2), we derive

$$\eta_i \int_{\Omega_i} |\nabla u|^2 dx + \eta_i \int_{\Omega_i} \bar{u} \Delta u dx + \eta_e \int_{\Omega_e} |\nabla u|^2 dx + \eta_e \int_{\Omega_e} \bar{u} \Delta u dx = 0. \tag{2.9}$$

Let $\gamma \neq 0$. Combining (2.9) and (2.1), we obtain

$$\frac{\alpha_i^2 - \gamma^2 + i2\alpha_i\gamma}{(\alpha_i^2 + \gamma^2)^2} \int_{\Omega_i} |\nabla u|^2 dx + \frac{1}{\alpha_e^2} \int_{\Omega_e} |\nabla u|^2 dx + \sigma^2 \int_{\mathbb{R}^2} |u|^2 dx = 0. \tag{2.10}$$

Equating the imaginary part of left-hand side in (2.10) to zero, we see that

$$\frac{2\alpha_i\gamma}{(\alpha_i^2 + \gamma^2)^2} \int_{\Omega_i} |\nabla u|^2 dx = 0. \tag{2.11}$$

The integral on the left-hand side in (2.11) is zero because the common factor is strictly positive or negative. Therefore, using (2.10), we establish that

$$\frac{1}{\alpha_e^2} \int_{\Omega_e} |\nabla u|^2 dx + \sigma^2 \int_{\mathbb{R}^2} |u|^2 dx = 0. \tag{2.12}$$

Then, the function u is zero on $\mathbb{R}^2$, since the factors before the integrals in (2.12) are positive.

Let $\gamma = 0$, then $\eta_i = 1/\alpha_i^2$. Therefore, it follows from (2.9) and (2.1) that

$$\frac{1}{\alpha_i^2}\int_{\Omega_i} |\nabla u|^2 dx + \frac{1}{\alpha_e^2}\int_{\Omega_e} |\nabla u|^2 dx + \sigma^2 \int_{\mathbb{R}^2} |u|^2 dx = 0. \tag{2.13}$$

Since the factors before the integrals in (2.13) are positive, the function u is identical zero on $\mathbb{R}^2$, which contradicts our initial assumption that u is an eigenfunction of problem (2.1)–(2.3), and the proof is completed. ∎

3. Müller boundary integral equations.

In this section, we reduce GCFEP to an eigenvalue problem for a set of BIEs.

LEMMA 3.1. *Let $\gamma \in \mathbb{R}$ be given. If u is an eigenfunction of problem* (2.1)–(2.3) *corresponding to an eigenvalue $k \in \mathbb{L}$, then*

$$u(x) = -\int_\Gamma \left(u^-(y)\frac{\partial G_i(k,\gamma;x,y)}{\partial n(y)} - G_i(k,\gamma;x,y)\frac{\partial u^-(y)}{\partial n(y)}\right) dl(y), \quad x \in \Omega_i, \tag{3.1}$$

$$u(x) = \int_\Gamma \left(u^+(y)\frac{\partial G_e(k;x,y)}{\partial n(y)} - G_e(k;x,y)\frac{\partial u^+(y)}{\partial n(y)}\right) dl(y), \quad x \in \Omega_e, \tag{3.2}$$

where $G_j(x,y) = \frac{i}{4}H_0^{(1)}(k_j|x-y|), \quad j = i,\ e.$

Proof. To prove (3.1), it is enough to apply on Ω_i the well known integral representation for solutions of the Helmholtz equation (see, e.g., [7, Theorem 3.1, p. 68]). The proof of (3.2) is similar. Indeed,

$$\begin{aligned} u(x) = &\int_\Gamma \left(u^+(y)\frac{\partial G_e(k;x,y)}{\partial n(y)} - G_e(k;x,y)\frac{\partial u^+(y)}{\partial n(y)}\right) dl(y) \\ &- \int_{\Gamma_R} \left(u^-(y)\frac{\partial G_e(k;x,y)}{\partial r(y)} - \frac{\partial u^-(y)}{\partial r(y)} G_e(k;x,y)\right) dl(y), \quad x \in \Omega_R \backslash \overline{\Omega}_i. \end{aligned} \tag{3.3}$$

Arguing as in the proof of Lemma 5.1 in [18], we see that the second integral in the right-hand side of (3.3) vanishes for any $k \in \mathbb{L}$, $\gamma \in \mathbb{R}$, and u satisfying (2.3). ∎

Using (2.2), we define the functions

$$u(x) = u^+(x) = u^-(x), \quad v(x) = \frac{\eta_e + \eta_i}{2\eta_i}\frac{\partial u^+(x)}{\partial n(x)} = \frac{\eta_e + \eta_i}{2\eta_e}\frac{\partial u^-(x)}{\partial n(x)}, \quad x \in \Gamma. \tag{3.4}$$

Adding term by term the limit values of the integral representations (3.1), (3.2), and their normal derivatives from both sides of the boundary Γ and using the well known properties of the potentials (see, e.g., [7, p. 47]), we derive

$$u(x) - \int_\Gamma K_{1,1}(k,\gamma;x,y)u(y)dl(y) - \int_\Gamma K_{1,2}(k,\gamma;x,y)v(y)dl(y) = 0, \quad x \in \Gamma, \tag{3.5}$$

$$v(x) - \int_\Gamma K_{2,1}(k,\gamma;x,y)u(y)dl(y) - \int_\Gamma K_{2,2}(k,\gamma;x,y)v(y)dl(y) = 0, \quad x \in \Gamma, \tag{3.6}$$

where the kernels are

$$K_{1,1}(k,\gamma;x,y) = \frac{\partial G_e(k;x,y)}{\partial n(y)} - \frac{\partial G_i(k,\gamma;x,y)}{\partial n(y)}, \quad x,y \in \Gamma, \tag{3.7}$$

$$K_{1,2}(k,\gamma;x,y) = \frac{2\eta_e}{\eta_e+\eta_i} G_i(k,\gamma;x,y) - \frac{2\eta_i}{\eta_e+\eta_i} G_e(k;x,y), \quad x,y \in \Gamma, \tag{3.8}$$

$$K_{2,1}(k,\gamma;x,y) = \frac{\partial^2 G_e(k;x,y)}{\partial n(x)\partial n(y)} - \frac{\partial^2 G_i(k,\gamma;x,y)}{\partial n(x)\partial n(y)}, \quad x,y \in \Gamma, \tag{3.9}$$

$$K_{2,2}(k,\gamma;x,y) = \frac{2\eta_e}{\eta_e+\eta_i} \frac{\partial G_i(k,\gamma;x,y)}{\partial n(x)} - \frac{2\eta_i}{\eta_e+\eta_i} \frac{\partial G_e(k;x,y)}{\partial n(x)}, \quad x,y \in \Gamma. \tag{3.10}$$

The set of integral equations (3.5), (3.6) is called the set of Müller boundary integral equations (BIEs) (see, e.g., [31]). The next lemma was proved in [36].

LEMMA 3.2. *For each $k \in \mathbb{L}$, $\gamma \in \mathbb{R}$, and $x \in \Gamma$ we have*

$$\lim_{y\to x} K_{1,1}(k,\gamma;x,y) = 0, \quad \lim_{y\to x} \frac{K_{1,2}(k,\gamma;x,y)}{\ln|x-y|} = \frac{(\eta_i-\eta_e)}{\pi(\eta_e+\eta_i)}, \tag{3.11}$$

$$\lim_{y\to x} \frac{K_{2,1}(k,\gamma;x,y)}{\ln|x-y|} = \frac{k_i^2-k_e^2}{4\pi}, \quad \lim_{y\to x} K_{2,2}(k,\gamma;x,y) = \frac{\xi(x)}{2\pi}\left(\frac{\eta_i-\eta_e}{\eta_e+\eta_i}\right), \tag{3.12}$$

where ξ is the curvature of the curve Γ.

4. Nonlinear eigenvalue problem for a holomorphic Fredholm operator-valued function.

By $C(\Gamma)$, we denote the Banach space of continuous functions on Γ with the usual maximum norm (see, e.g., [21, p. 3])

$$||u||_\infty = \max_{x\in\Gamma} |u(x)|. \tag{4.1}$$

We introduce the following integral operators with kernels defined in (3.7)–(3.10):

$$\left(B_{i,j}(k,\gamma)w_j\right)(x) = \int_\Gamma K_{i,j}(k,\gamma;x,y)w_j(y)dl(y), \quad x \in \Gamma, \tag{4.2}$$

where $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}$. It follows from Lemma 3.2 that (see, e.g., [21, Theorem 2.8, p. 17, and Problem 2.3, p. 27]) for each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}$ the operators $B_{i,j} : C(\Gamma) \to C(\Gamma)$ are bounded with

$$||B_{i,j}(k,\gamma)||_\infty = \max_{x\in\Gamma} \int_\Gamma |K_{i,j}(k,\gamma;x,y)|dl(y), \quad i,j = 1,2. \tag{4.3}$$

Moreover, these integral operators are compact (see, e.g., [21, Theorem 2.23, p. 26]). Therefore, the next theorem is true.

THEOREM 4.1. *For each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}$ the integral operator $B : W \to W$, where $W = C(\Gamma) \times C(\Gamma)$, defined by*

$$B(k,\gamma)w = \begin{bmatrix} B_{1,1}(k,\gamma) & B_{1,2}(k,\gamma) \\ B_{2,1}(k,\gamma) & B_{2,2}(k,\gamma) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \quad w = (u,v)^T, \quad u,v \in C(\Gamma), \tag{4.4}$$

is compact.

Let us rewrite set (3.5), (3.6) in the form

$$w = B(k,\gamma)w. \tag{4.5}$$

We are interested in finding $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}$ such that there exist nonzero solutions $w \in W$ of (4.5).

Theorem 4.2. *Assume that $\gamma \in \mathbb{R}$ is given. If $u \in U$ is an eigenfunction of problem* (2.1)–(2.3) *corresponding to an eigenvalue $k \in \mathbb{L}$, then the function*

$$w = \left(u^+, \frac{\eta_e + \eta_i}{2\eta_i}\frac{\partial u^+}{\partial n}\right)^T = \left(u^-, \frac{\eta_e + \eta_i}{2\eta_e}\frac{\partial u^-}{\partial n}\right)^T, \tag{4.6}$$

belongs to W, and w is a nontrivial solution of (4.5) *with the same values of k and γ.*

Proof. The first assertion follows immediately from Lemma 3.1. Indeed, any eigenfunction of (2.1)–(2.3) is continuous on $\bar{\Omega}_i$ and $\bar{\Omega}_e$, hence u^+ and u^- belong to $C(\Gamma)$. Further, $\partial u^+/\partial n$ and $\partial u^-/\partial n$ defined in (2.4) belong to $C(\Gamma)$ as the uniform limits of the continuous functions. Clearly, w defined in (4.6) is the solution of (4.5) by construction of this equation. If we assume that $w = 0$, then using (3.1) and (3.2), we see that $u = 0$, $x \in \mathbb{R}^2$, which contradicts the assumption that u is an eigenfunction of GCFEP (2.1)–(2.3). ∎

The assertion in the opposite direction relative to the statement of Theorem 4.2 is not true (see, e.g., [22]). The nonequivalence of the eigenvalue problems (2.1)–(2.3) and (4.5) implies that, in general, the Müller boundary integral formulation (4.5) exhibits additional complex eigenvalues which are not eigenvalues of (2.1)–(2.3). Paper [22] and references therein suggest some ways how these additional complex eigenvalues can be identified and removed. In practice, we use Müller BIEs only for numerical solving LEP with real-valued eigenvalues and do not observe spurious modes.

Let us choose a fixed $\gamma \in \mathbb{R}$ and put $A(k) = I - B(k)$, where $I : W \to W$ is the identity operator, and $B(k) : W \to W$ is defined in (4.4). For each $k \in \mathbb{L}$ the operator $B(k)$ is compact. A nonzero vector $w \in W$ is called an eigenvector of the operator-valued function $A(k)$ corresponding to an eigenvalue $k \in \mathbb{L}$ if

$$A(k)w = (I - B(k))w = 0 \tag{4.7}$$

is satisfied. Denote by $\mathcal{L}(W, W)$ the space of all bounded linear operators acting in W. The set $\sigma(A)$ of all $k \in \mathbb{L}$, for which the operator $A(k)$ does not have a bounded inverse operator on W, is called the spectrum of $A(k)$, and $\rho(A) = \{k \in \mathbb{L} : A(k)^{-1} \in \mathcal{L}(W, W)\}$ is called the resolvent set of $A(k)$.

The operator-valued function $B(k) \in \mathcal{L}(W, W)$ is called holomorphic in $k \in \mathbb{L}$ if it can be represented as the sum of a power series

$$B(k) = \sum_{p=0}^{\infty} B_p (k - k_0)^p, \quad B_p \in \mathcal{L}(W, W),$$

which is convergent in $\mathcal{L}(W, W)$ in a neighborhood of every point $k_0 \in \mathbb{L}$.

Theorem 4.3. *For each $\gamma \in \mathbb{R}$ the operator-valued function $B(k)$ is holomorphic in $k \in \mathbb{L}$.*

Proof. We prove that the operator-valued function $B(k)$ is holomorphic in $k \in \mathbb{L}$ for each $\gamma \in \mathbb{R}$ following [27] (see also [19, p. 365]). This is true if and only if for each $w \in W$ and for each bounded linear functional g defined on W, the function $g[B(k)w]$ is holomorphic in $k \in \mathbb{L}$. Let us recall that the kernels $K_{i,j}$, $i, j = 1, 2$, are holomorphic functions in $k \in \mathbb{L}$ for each $\gamma \in \mathbb{R}$,

$$K_{i,j}(k; x, y) = \sum_{p=0}^{\infty} K_{i,j;k}^{(p)}(k_0; x, y) \frac{(k - k_0)^p}{p!}, \quad |k - k_0| < \varepsilon_{i,j}, \quad k_0 \in \mathbb{L}. \tag{4.8}$$

Here, $K^{(p)}_{i,j;k}$ is the pth derivative of the function $K_{i,j}$ with respect to the variable k. Hence,

$$(4.9)\quad (B_{i,j}(k)w_j)(x) = \int_\Gamma K_{i,j}(k;x,y)w_j(y)dl(y)$$
$$= \int_\Gamma \sum_{p=0}^{\infty} K^{(p)}_{i,j;k}(k_0;x,y)\frac{(k-k_0)^p}{p!}w_j(y)dl(y)$$
$$= \sum_{p=0}^{\infty}(k-k_0)^p \int_\Gamma \frac{1}{p!}K^{(p)}_{i,j;k}(k_0;x,y)w_j(y)dl(y), \quad w=(w_1,w_2).$$

Arguing as in the proofs of Lemmas 1–4 in [36], we see that

$$(4.10)\qquad \lim_{|x-y|\to 0} K^{(p)}_{i,i;k}(k_0;x,y) = 0, \quad i=1,2, \quad p=1,2,\dots,$$

$$(4.11)\qquad \lim_{|x-y|\to 0} \frac{1}{p!}K^{(p)}_{1,2;k}(k_0;x,y) = \frac{(-1)^p(\eta_e-\eta_i)}{\pi k_0^p(\eta_e+\eta_i)p}, \quad p=1,2,\dots,$$

$$(4.12)\quad \lim_{|x-y|\to 0}\frac{K^{(1)}_{2,1;k}(k_0;x,y)}{\ln|x-y|} = \frac{k_0\left(\nu_i^2-\nu_e^2\right)}{2\pi}, \quad \lim_{|x-y|\to 0}\frac{K^{(2)}_{2,1;k}(k_0;x,y)}{\ln|x-y|} = \frac{\nu_i^2-\nu_e^2}{2\pi},$$

$$(4.13)\qquad \lim_{|x-y|\to 0} \frac{1}{p!}K^{(p)}_{2,1;k}(k_0;x,y) = \frac{(-1)^{p-3}\left(\nu_i^2-\nu_e^2\right)}{2\pi k_0^{p-2}p(p-1)(p-2)}, \quad p=3,4,\dots,$$

for all $k_0\in\mathbb{L}$, $x,y\in\Gamma$. Therefore, combining (4.10)–(4.13) with asymptotic expansions of the Bessel functions of large orders (see, e.g., [1, p. 365]), we see that in (4.9) the operators $B^{(p)}_{i,j}: C(\Gamma)\to C(\Gamma)$, defined by

$$(4.14)\qquad \left(B^{(p)}_{i,j;k}(k_0)w_j\right)(x) = \int_\Gamma \frac{1}{p!}K^{(p)}_{i,j;k}(k_0;x,y)w_j(y)dl(y), \quad x\in\Gamma, \quad k_0\in\mathbb{L},$$

are uniformly bounded in the parameter p, hence the operator

$$(4.15)\qquad B^{(p)}w = \begin{bmatrix} B^{(p)}_{1,1} & B^{(p)}_{1,2} \\ B^{(p)}_{2,1} & B^{(p)}_{2,2}\end{bmatrix}\begin{bmatrix} w_1 \\ w_2\end{bmatrix}$$

is uniformly bounded in p. The functional g is continuous. Therefore, for each $k_0\in\mathbb{L}$ and $w\in W$ there exists $\varepsilon>0$ such that

$$(4.16)\qquad g\left[B(k)w\right] = \sum_{p=0}^{\infty} b_p(k_0)(k-k_0)^p, \quad |k-k_0|<\varepsilon,$$

where $b_p(k_0) = g\left[B^{(p)}(k_0)w\right]$. Using the Weierstrass M-test, we see that the functional series on the right-hand side of (4.16) converges absolutely and uniformly, since the majorizing number series is a geometric progression. Equality (4.16) means that the function $g[B(k)w]$ is holomorphic in $k\in\mathbb{L}$. ∎

Now we investigate spectral properties of $A(k)$ following the Appendix in [20]. Here we recall that a value $\gamma\in\mathbb{R}$ is fixed. Let k_0 be an eigenvalue of $A(k)$. The dimension of $\ker A(k_0)$ is called the geometric multiplicity of k_0. Let w_0 be an eigenvector corresponding to k_0. The elements $w_1,\dots,w_{m-1}$ in W are called generalized eigenvectors if $\sum_{j=0}^{n}(j!)^{-1}A^{(j)}(k_0)w_{n-j}=0$, $n=1,\dots,m-1$, where $A^{(j)}(k)=dA(k)/dk^j$.

It is said that the ordered collection $w_0, w_1, \dots, w_{m-1}$ is a Jordan chain corresponding to k_0. By $S(A, k_0)$, we denote the set of all vector functions $\Phi(k)$ represented in the form of $\Phi(k) = \sum_{j=0}^{m-1} (k - k_0)^{j-m} w_j$, where $\{w_j\}_{j=0}^{m-1}$ is a Jordan chain and $m \geq 1$, that satisfies $A(k)\Phi(k) = O(1)$ for small enough $|k - k_0|$. The dimension of $S(A, k_0)$ is called the algebraic multiplicity of k_0. If w_0 is an eigenvector of $A(k)$ corresponding to k_0, then $w_0(k - k_0)^{-1} \in S(A, k_0)$. Therefore, the geometric multiplicity does not exceed the algebraic multiplicity.

Theorem 4.4. *The following statements are true.*

1. *For each $\gamma \in \mathbb{R}$, the resolvent set of the operator-valued function $A(k)$ is not empty, namely, $\mathbb{I}_+ \subset \rho(A)$.*
2. *For each $\gamma \in \mathbb{R}$, the spectrum $\sigma(A)$ of the operator-valued function $A(k)$ can be only a set of isolated points on $\mathbb{L}$, which are the eigenvalues of $A(k)$ of finite algebraic multiplicities.*
3. *Each eigenvalue k of the operator-valued function $A(k)$ depends continuously on $\gamma \in \mathbb{R}$ and can appear and disappear only on the boundary of its analyticity domain, i.e., at zero and at infinity on $\mathbb{L}$.*

Proof. Arguing as in the proof of [7, Theorem 3.41, p. 101], we see that for all $k \in \mathbb{I}_+$ and $\gamma \in \mathbb{R}$ problem (4.5) has only the trivial solution. Therefore, using the compactness of the operator $B(k)$ (see Theorem 4.1) and the Fredholm alternative (see, e.g, [21, p. 47]), we see that the operator $A(k)$ has a bounded inverse operator in W for all $k \in \mathbb{I}_+$ and $\gamma \in \mathbb{R}$. Now the second assertion follows from Theorem 4.3 on the holomorphicity of the operator-valued function $B(k)$ and Proposition A.8.4. [20, p. 422]. Arguing as in the proof of Theorem 4.3, we see that for each $k \in \mathbb{L}$ the operator-valued function $B(k, \gamma)$ is real-holomorphic in $\gamma \in \mathbb{R}$ (see, e.g., [19, p. 365]). Combining this with holomorphicity of $B(k)$ in $k \in \mathbb{L}$ for each $\gamma \in \mathbb{R}$, we see that $B(k, \gamma)$ is jointly continuous in (k, γ) for each $(k, \gamma) \in \mathbb{L} \times \mathbb{R}$ (see also [2, Proposition 6.1, p. 1148]). Thus assertion 3 follows from [38, Theorem 3]. ∎

The first two statements of Theorem 4.4 for $\gamma \leq 0$ correspond to CFEP, while the next proposition following from Theorem 4.4 describes the spectrum of LEP.

Corollary 4.5. *If for some $\gamma > 0$ the intersection of the spectrum $\sigma(A)$ and the positive real semiaxis of $\mathbb{L}_0$ is not empty, then it can be only a set of isolated points $k > 0$, which are the eigenvalues of $A(k)$ of finite algebraic multiplicities.*

5. Nyström method.

In this section, following [31], we present the Nyström method for numerical solution of problem (4.7). We assume that the contour Γ is 2-times differentiable and has the parameterization $r(t) = (r_1(t),\ r_2(t))$, where $t \in [0, 2\pi]$. Let us recall that the kernels $K_{1,1}$ and $K_{2,2}$ are the smooth functions if $\eta_i = \eta_e$, then the kernel $K_{1,2}$ is smooth, otherwise $K_{1,2}$ has the logarithmic singularity, and $K_{2,1}$ always has the logarithmic singularity. Still, for uniformity, we write all these functions in the form (see [6, p. 69])

$$K_{i,j}(t, \tau) = Q_{i,j}(t, \tau) \ln\left(4\sin^2\frac{t - \tau}{2}\right) + P_{i,j}(t, \tau), \quad i, j = 1, 2, \tag{5.1}$$

where $Q_{i,j}(t, \tau)$ and $P_{i,j}(t, \tau)$ are continuous functions on $[0, 2\pi] \times [0, 2\pi]$.

Let $\Xi_n = \{t_j\}_{j=0}^{2n-1}$ be a uniform grid on $[0, 2\pi]$ with the mesh size $h = \pi/n$, i.e., $t_j = jh$, $j = 0, \dots, 2n - 1$, $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. We are looking for approximate solutions of problem (4.7) in the form

$$u^{(n)}(t) = \sum_{j=0}^{2n-1} \Big(R_j^{(n)}(t)Q_{1,1}(t,t_j) + \frac{\pi}{n}P_{1,1}(t,t_j)\Big)|r'(t_j)|u_j + \sum_{j=0}^{2n-1} \Big(R_j^{(n)}(t)Q_{1,2}(t,t_j) + \frac{\pi}{n}P_{1,2}(t,t_j)\Big)|r'(t_j)|v_j, \tag{5.2}$$

$$v^{(n)}(t) = \sum_{j=0}^{2n-1} \Big(R_j^{(n)}(t)Q_{2,1}(t,t_j) + \frac{\pi}{n}P_{2,1}(t,t_j)\Big)|r'(t_j)|u_j + \sum_{j=0}^{2n-1} \Big(R_j^{(n)}(t)Q_{2,2}(t,t_j) + \frac{\pi}{n}P_{2,2}(t,t_j)\Big)|r'(t_j)|v_j, \tag{5.3}$$

$$R_j^{(n)}(t) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j) - \frac{\pi}{n^2}\cos n(t-t_j), \quad j = 0,\dots,2n-1,$$

$u_i = u(t_i)$, $v_i = v(t_i)$, $i = 0,\dots,2n-1$. In (5.2), (5.3) we use the trapezoidal rule for approximation of integrals with continuous kernels $P_{i,j}(t,\tau)|r'(\tau)|$, $i,j = 1,2$, and the special quadrature rule for integrals with the kernels of the form (see [6, p. 69])

$$Q_{i,j}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)|r'(\tau)|, \quad i,j = 1,2.$$

These numerical integration formulas were obtained in [6] by replacing the integrands with their trigonometric interpolation polynomials and then integrating exactly.

Equating the left-hand sides with the right-hand sides in (5.2) and (5.3) at the grid points, we obtain the following set of linear equations for the unknown values u_j and v_j, $j = 0,\dots,2n-1$:

$$u_i - \sum_{j=0}^{2n-1} \Big(R_{|i-j|}^{(n)}Q_{1,1}(t_i,t_j) + \frac{\pi}{n}P_{1,1}(t_i,t_j)\Big)|r'(t_j)|u_j - \sum_{j=0}^{2n-1} \Big(R_{|i-j|}^{(n)}Q_{1,2}(t_i,t_j) + \frac{\pi}{n}P_{1,2}(t_i,t_j)\Big)|r'(t_j)|v_j = 0, \tag{5.4}$$

$$v_i - \sum_{j=0}^{2n-1} \Big(R_{|i-j|}^{(n)}Q_{2,1}(t_i,t_j) + \frac{\pi}{n}P_{2,1}(t_i,t_j)\Big)|r'(t_j)|u_j - \sum_{j=0}^{2n-1} \Big(R_{|i-j|}^{(n)}Q_{2,2}(t_i,t_j) + \frac{\pi}{n}P_{2,2}(t_i,t_j)\Big)|r'(t_j)|v_j = 0, \tag{5.5}$$

where $i = 0,\dots,2n-1$ and

$$R_j^{(n)} = R_j^{(n)}(0) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos\frac{mj\pi}{n} - \frac{(-1)^j\pi}{n^2}, \quad j = 0,\dots,2n-1.$$

The solutions of the nonlinear algebraic eigenvalue problem

$$A_n(k,\gamma)w_n = (I - B_n(k,\gamma))w_n = 0 \tag{5.6}$$

are approximations of solutions to (4.7) by the Nyström method. Here, $A_n(k,\gamma)$ is the matrix of the linear system (5.4), (5.5) with elements nonlinearly depending on k and γ. By $w_n = (u_n, v_n)^T$ we denote the vector of unknowns of this system. For a given $\gamma \in \mathbb{R}$, we denote by $\sigma(A_n)$ the spectrum of the matrix-valued function $A_n(k)$, and by $\rho(A_n)$ its resolvent set.

THEOREM 5.1. *For any given $\gamma \in \mathbb{R}$, the following statements are true.*

1. *For every eigenvalue k_0 of $A(k)$, there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ converging to k_0 with the eigenvalues k_n of $A_n(k)$.*
2. *If $\{k_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are some sequences of eigenvalues k_n of $A_n(k)$ and normalized eigenfunctions w_n of $A_n(k)$, so that $k_n \to k_0 \in \mathbb{L}$ $(n \in \mathbb{N})$, then*
   (i) *k_0 is an eigenvalue of $A(k)$,*
   (ii) *$\{w_n\}_{n\in\mathbb{N}}$ is a discretely compact sequence and its cluster points are normalized eigenfunctions of $A(k_0)$.*
3. *For every compact $L_0 \subset \rho(A)$, the sequence $\{A_n(k)\}_{n\in\mathbb{N}}$ is stable on L_0, i.e., there exist $n(L_0)$ and $c(L_0)$ such that $L_0 \subset \rho(A_n)$, $\|A_n(k)^{-1}\| \le c(L_0)$ for all $k \in L_0$ and $n \ge n(L_0)$.*

We now explain the meaning of the term "discretely compact sequence." The proof of this theorem is based on the general results of the discrete convergence theory [41] applied for investigation of approximate methods in the eigenvalue problem where the parameter appears nonlinearly [16]. Below are some definitions and results of [16], [41].

In this section, we consider the operator $A(k)$ for each $k \in \mathbb{L}$ as an operator in the space $W = C_{2\pi} \times C_{2\pi}$, where $C_{2\pi}$ is the space of 2π-periodic continuous functions with the standard maximum norm,

$$||u||_{C_{2\pi}} = \max_{t\in[0,2\pi]} |u(t)|, \quad u \in C_{2\pi}.$$

We introduce the space C^{2n} of grid functions defined on Ξ_n with the norm $||u_n||_{C^{2n}} = \max_{0\le j\le 2n-1} |u_n(t_j)|$, $u_n \in C^{2n}$. For each given $k \in \mathbb{L}$, the matrix $A_n(k)$ defines the operator in $W_n = C^{2n} \times C^{2n}$. Following [41, p. 52], we define the family of the connection operators $p_n : W \to W_n$ as the operators restricting functions $w \in W$ to the grid Ξ_n: $p_n w \in W_n$ is the grid function with the values $(p_n w)(t_j) = w(t_j)$, $j = 0, \ldots, 2n-1$. In [41, p. 52], it was proved that the operator p_n belongs to the space of bounded linear operators $\mathcal{L}(W, W_n)$,

$$\|p_n\|_{W\to W_n} = 1, \tag{5.7}$$

and $||p_n w||_{W_n} \to ||w||_W$, $n \to \infty$, for all $w \in W$. By $\mathbb{N}', \mathbb{N}'', \mathbb{N}''', \ldots$, we denote infinite subsequences of $\mathbb{N}$. The sequence $\{w_n\}_{n\in\mathbb{N}'}$ of elements $w_n \in W_n$ is discretely converging to the element $w \in W$ if $||w_n - p_n w||_{W_n} \to 0$ for $n \in \mathbb{N}'$; we will write $w_n - \to w$ $(n \in \mathbb{N}')$. The sequence $\{w_n\}_{n\in\mathbb{N}'}$ of elements $w_n \in W_n$ is discretely compact if for every subsequence $\{w_n\}_{n\in\mathbb{N}''}$, $\mathbb{N}'' \subseteq \mathbb{N}'$, there exist $\mathbb{N}''' \subseteq \mathbb{N}''$ and $w \in W$ such that $w_n - \to w$ $(n \in \mathbb{N}''')$. Assume that the operator $A \in \mathcal{L}(W, W)$ and a sequence $A_n \in \mathcal{L}(W_n, W_n)$ is given. Then, by definition, $\{A_n\}_{n\in\mathbb{N}'}$ approximates A if $||A_n p_n w - p_n A w||_{W_n} \to 0$ $(n \in \mathbb{N}')$ for all $w \in W$. We also say that $\{A_n\}_{n\in\mathbb{N}'}$ is regular if $||w_n||_{W_n} \le 1$ $(n \in \mathbb{N}')$ together with the fact that $\{A_n w_n\}_{n\in\mathbb{N}'}$ is discretely compact implies that the sequence $\{w_n\}_{n\in\mathbb{N}'}$ is discretely compact.

Proof of Theorem 5.1. The operator-valued functions $A(k)$ and $A_n(k)$ are holomorphic in $k \in \mathbb{L}$. The statement for $A(k)$ was proved in Theorem 4.3, the statement for $A_n(k)$ is proved similarly. For each $k \in \mathbb{L}$ the operator $B(k)$ is compact (see Theorem 4.1) and $B_n(k)$ is finite-dimensional. As was proved in Theorem 4.4, the resolvent set $\rho(A)$ is not empty, namely, $\mathbb{I}_+ \subset \rho(A)$. Therefore, to prove Theorem 5.1, we should check the conditions $(b3)$–$(b5)$ of [16, Theorem 2]. We note here that conditions $(b3)$–$(b5)$ mean that the sequence $\{A_n(k)\}_{n\in\mathbb{N}}$ of the holomorphic Fredholm operator-valued functions $A_n(k)$ regularly approximates $A(k)$ on $\mathbb{L}$.

$(b3)$ The sequence $\{A_n(k)\}_{n\in\mathbb{N}}$ is uniformly bounded on every compact $L_0 \subset \mathbb{L}$, i.e., for each compact $L_0 \subset \mathbb{L}$ there exists c such that $\|A_n(k)\|_{W_n\to W_n} \le c$ for all $k \in L_0$, $n \in \mathbb{N}$. Now we prove this statement. Let us recall that $A_n(k) = I - B_n(k)$, $k \in \mathbb{L}$. Here,

$$B_n(k) = \begin{bmatrix} B_n^{(1,1)}(k) & B_n^{(1,2)}(k) \\ B_n^{(2,1)}(k) & B_n^{(2,2)}(k) \end{bmatrix} \tag{5.8}$$

is the 2-by-2 block matrix-valued function with the entries

$$\left(B_n^{(l,m)}(k)\right)_{i,j} = \left(R_{|i-j|}^{(n)} Q_{l,m}(k,t_i,t_j) + \frac{\pi}{n} P_{l,m}(k,t_i,t_j)\right) |r'(t_j)|, \tag{5.9}$$

where $i, j = 0, \ldots, 2n-1$, $l, m = 1, 2$. Clearly, we have the following inequality:

$$\|A_n(k)\|_{W_n\to W_n} \le 1 + 4 \max_{l,m=1,2} \left\| B_n^{(l,m)}(k) \right\|_{C^{2n}\to C^{2n}}. \tag{5.10}$$

Now we estimate a single element $B_n^{(l,m)}(k)$ of the block matrix $B_n(k)$:

$$\begin{aligned} \left\| B_n^{(l,m)}(k) \right\|_{C^{2n}\to C^{2n}} \le{}& \max_{0\le i\le 2n-1} \sum_{j=0}^{2n-1} \left| R_{|i-j|}^{(n)} Q_{l,m}(k,t_i,t_j) \right| |r'(t_j)| \\ &+ \max_{0\le i\le 2n-1} \sum_{j=0}^{2n-1} \frac{\pi}{n} |P_{l,m}(k,t_i,t_j)| \, |r'(t_j)|. \end{aligned} \tag{5.11}$$

For each term of (5.11), we obtain the estimates

$$\begin{aligned} &\max_{0\le i\le 2n-1} \sum_{j=0}^{2n-1} \left| R_{|i-j|}^{(n)} Q_{l,m}(k,t_i,t_j) \right| |r'(t_j)| \\ &\quad\le \max_{t,\tau\in[0,2\pi]} |Q_{l,m}(k,t,\tau)| \, |r'(\tau)| \max_{0\le i\le 2n-1} \sum_{j=0}^{2n-1} \left| R_{|i-j|}^{(n)} \right|, \end{aligned} \tag{5.12}$$

$$\max_{0\le i\le 2n-1} \sum_{j=0}^{2n-1} \frac{\pi}{n} |P_{l,m}(k,t_i,t_j)| \, |r'(t_j)| \le 2\pi \max_{t,\tau\in[0,2\pi]} |P_{l,m}(k,t,\tau)| \, |r'(\tau)|. \tag{5.13}$$

Using the estimate (see [21, p. 209]) $\max_{t\in[0,2\pi]} \sum_{j=0}^{2n-1} |R_j^{(n)}(t)| \le \sqrt{2}\pi^2$, we obtain

$$\max_{0\le i\le 2n-1} \sum_{j=0}^{2n-1} |R_{|i-j|}^{(n)}| \le \sqrt{2}\pi^2. \tag{5.14}$$

Finally, using (5.10)–(5.14), we see that $\|A_n(k)\|_{W_n\to W_n} \le c(k)$, where $c(k)$ is the continuous function on $\mathbb{L}$ of the form

$$\begin{aligned} c(k) ={}& 4 \max_{i,j=1,2} \left(\sqrt{2}\pi^2 \max_{t,\tau\in[0,2\pi]} |Q_{i,j}(k,t,\tau)| \, |r'(\tau)| + 2\pi \max_{t,\tau\in[0,2\pi]} |P_{i,j}(k,t,\tau)| \, |r'(\tau)| \right) \\ &+ 1. \end{aligned}$$

Thus we see that condition ($b3$) holds true.

($b4$) The sequence $\{A_n(k)\}_{n\in\mathbb{N}}$ approximates $A(k)$ for every $k \in \mathbb{L}$. Let us prove this statement. To simplify the notation, we do not always indicate the dependence of the operators on the parameter k. We introduce the operator $B^{(n)} : W_n \to W$, using the right-hand sides of (5.2) and (5.3), namely,

$$B^{(n)} w_n = \begin{bmatrix} B^{(n)}_{1,1} & B^{(n)}_{1,2} \\ B^{(n)}_{2,1} & B^{(n)}_{2,2} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \tag{5.15}$$

$$B^{(n)}_{l,m} w_{m,n} = \sum_{j=0}^{2n-1} \left(R^{(n)}_j(t) Q_{l,m}(t,t_j) + \frac{\pi}{n} P_{l,m}(t,t_j) \right) |r'(t_j)| w_{m,n}(t_j),$$

where $t \in [0, 2\pi]$; $l, m = 1, 2$; $w_{1,n} = u_n$, $w_{2,n} = v_n$. It follows from (5.4), (5.5), and (5.15) that $A_n p_n w = p_n w - p_n B^{(n)} p_n w$ for all $w \in W$. Therefore,

$$A_n p_n w - p_n A w = p_n w - p_n B^{(n)} p_n w - p_n w + p_n B w = p_n B w - p_n B^{(n)} p_n w \tag{5.16}$$

and

$$\|A_n p_n w - p_n A w\|_{W_n} \le \|p_n\|_{W\to W_n} \left\| B w - B^{(n)} p_n w \right\|_W. \tag{5.17}$$

Using [21, Theorems 12.8, 12.13, pp. 202–209], for $l, m = 1, 2$ and $u \in C_{2\pi}$, we obtain

$$\max_{t\in[0,2\pi]} \left| \int_0^{2\pi} P_{l,m}(t,\tau)|r'(\tau)|u(\tau)d\tau - \sum_{j=0}^{2n-1} \frac{\pi}{n} P_{l,m}(t,t_j)|r'(t_j)|u(t_j) \right| \to 0, \quad n \to \infty,$$

$$\max_{t\in[0,2\pi]} \left| \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) Q_{l,m}(t,\tau)|r'(\tau)|u(\tau)d\tau - \sum_{j=0}^{2n-1} R^{(n)}_j(t) Q_{l,m}(t,t_j)|r'(t_j)|u(t_j) \right| \to 0, \quad n \to \infty.$$

Therefore,

$$\left\| B w - B^{(n)} p_n w \right\|_W \to 0, \quad n \to \infty. \tag{5.18}$$

Finally, combining (5.17), (5.18), and (5.7), we see that condition ($b4$) is true,

$$\|A_n p_n w - p_n A w\|_{W_n} \to 0 \quad (n \in \mathbb{N}) \quad \text{for each} \quad w \in W.$$

($b5$) The sequence $\{A_n(k)\}_{n\in\mathbb{N}}$ is regular for every $k \in \mathbb{L}$. Indeed, if we assume that the sequence of vectors $\{A_n w_n\}_{n\in\mathbb{N}}$ is discretely compact, then, by definition, for any subset $\mathbb{N}'$ of the set of $\mathbb{N}$ there exists $\mathbb{N}'' \subseteq \mathbb{N}'$ such that the sequence $\{A_n w_n = w_n - B_n w_n\}_{n\in\mathbb{N}''}$ discretely converges to a function $f \in W$. Using [21, Theorems 12.8, 12.13, pp. 202–209], and arguing as in the proof of statement ($b4$), we see that the operators $B^{(n)} p_n : W \to W$ are collectively compact. Denote by $w^{(n)}$ the piecewise linear interpolation polynomial for w_n, so that

$$w_n = p_n w^{(n)}. \tag{5.19}$$

Therefore, $\|w_n\|_{W_n} = \left\|w^{(n)}\right\|_W$. Hence, $\|w_n\|_{W_n} \le 1$ $(n \in \mathbb{N})$ implies $\left\|w^{(n)}\right\|_W \le 1$ for $n \in \mathbb{N}''$. Whence (see, e.g., [21, p. 7]) the sequence $\{B^{(n)} p_n w^{(n)}\}_{n\in\mathbb{N}''}$ contains a convergent subsequence $\{B^{(n)} p_n w^{(n)}\}_{n\in\mathbb{N}'''}$, $\mathbb{N}''' \subseteq \mathbb{N}''$,

$$\left\|B^{(n)} p_n w^{(n)} - g\right\|_W \to 0 \quad (n \in \mathbb{N}''') \quad g \in W. \tag{5.20}$$

It follows from (5.19) and (5.15) that $B_n w_n = p_n B^{(n)} p_n w^{(n)}$. Therefore, the following inequality is true:

$$\|B_n w_n - p_n g\|_{W_n} = \left\|p_n B^{(n)} p_n w^{(n)} - p_n g\right\|_{W_n} \le \|p_n\|_{W\to W_n} \|B^{(n)} p_n w^{(n)} - g\|_W. \tag{5.21}$$

Thus, combining (5.21) with (5.20) and (5.7), we finally get $\|B_n w_n - p_n g\|_{W_n} \to 0$ $(n \in \mathbb{N}''')$, where $g \in W$, this means that the sequence $\{B_n w_n\}_{n\in\mathbb{N}'''}$ discretely converges to $g \in W$. Consequently, the sequence $\{w_n\}_{n\in\mathbb{N}'''}$ discretely converges to $w = f + g \in W$, and condition $(b5)$ is valid. □

The operator p_n is linear. Therefore, the next theorem follows from [17, Theorem 2 and Remark 1, pp. 394–395].

THEOREM 5.2. *Assume that $\gamma \in \mathbb{R}$ is given, k_0 is an eigenvalue of $A(k)$, and $L_0 \subset \mathbb{L}$ is a compact set with the boundary $\Gamma_0 \subset \rho(A)$ so that $L_0 \cap \sigma(A) = \{k_0\}$. Let us denote by ε_n the maximum of the approximation error over $k \in \Gamma_0$ and $w \in G(A, k_0)$,*

$$\varepsilon_n = \sup\{\|A_n(k) p_n w - p_n A(k) w\|_{W_n} : \ k \in \Gamma_0, \ w \in G(A, k_0), \ \|w\|_W = 1\}. \tag{5.22}$$

Here, $G(A, k_0)$ is the generalized eigenspace, i.e., the closed linear hull of all the generalized eigenfunctions of $A(k)$ corresponding to k_0. Then $\varepsilon_n \to 0$ $(n \in \mathbb{N})$ and the following estimations hold for almost all $n \in \mathbb{N}$:

(i) *$|k_n - k_0| \le c\varepsilon_n^{1/\kappa}$ for all $k_n \in \sigma(A_n) \cap L_0$, where $\kappa = \kappa(k_0, A)$ is the order of the pole k_0 of the operator-valued function $A^{-1}(k)$;*

(ii) *$|\bar{k}_n - k_0| \le c\varepsilon_n$, where $\bar{k}_n$ is the weighted (proportionally to their algebraic multiplicities) mean of all the eigenvalues of $A_n(k)$ in L_0, $\bar{k}_n = \sum_{k\in\sigma(A_n)\cap L_0} \mu_k \cdot k$, $\mu_k = \nu(k, A_n)/\nu(k, A)$, where $\nu(\cdot,\cdot)$ is the algebraic multiplicity of the corresponding eigenvalue k;*

(iii) *$\max\{|k_n - k_0| : \ k_n \in \sigma(A_n) \cap L_0\} \le c\varepsilon_n^{1/l_n}$, where l_n is the number of different eigenvalues of $A_n(k)$ in L_0.*

6. Numerical results.

In this section, we present numerical results only for LEP. The special case of the problem where k is real and positive is the most important in laser physics. In numerical examples presented further, we solve the discretized nonlinear eigenvalue problem using the residual inverse iteration algorithm (see [33] for its detailed description, which includes a method for calculating appropriate initial guess values). In all our computations, we assume that $k, \gamma > 0$ and the microcavity material has the refractive index $\alpha_i = 2.63$ (this is the effective index for a GaAs slab of 200 nm thickness in the infrared range) while the environment is air with $\alpha_e = 1$, and hence $k_e = k$. We consider only the H-polarized modes because in thin cavities their effective refractive index is lower than for the E-polarized modes [30].

For uniformly active microcavity shaped as a square with the side $2a$, we use a smooth approximation of Γ with the aid of the "supercircle" characterized by the parametric equation

$$r(t) = af(t)(\cos t, \sin t), \quad f(t) = \left((\cos t)^{2p} + (\sin t)^{2p}\right)^{-\frac{1}{2p}}, \quad t \in [0, 2\pi], \tag{6.1}$$

with parameter $p = 1$ corresponding to the circle and $p \to \infty$ to the square. Following [3], [4], we compute the function $a/|r''(\pi/4)|$ of the parameter p since its value should be at least 5–10 times less than $2\pi/(\kappa\alpha_i)$, where $\kappa = ka$. We see that for $\kappa \approx [1, 10]$ it is enough to choose $p = 10$. In all our computations we use this value of p. As we have found, taking p larger (i.e., making the corners sharper) results in the change of LEP eigenvalues in the 4th digit.

Observing the left-hand panels in Figures 2 and 3, we see that our algorithm converges exponentially in terms of the relative error $\varepsilon_r = \|(\kappa_n, \gamma_n) - (\tilde{\kappa}, \tilde{\gamma})\|_2 / \|(\tilde{\kappa}, \tilde{\gamma})\|_2$. Here, (κ_n, γ_n) are approximate values computed for different p and different order n of the interpolation polynomials. By $(\tilde{\kappa}, \tilde{\gamma})$, we denote the approximate values computed for maximum possible order n of the interpolation polynomials or exact solutions for the circular microcavity. We see also that the method converges faster for smaller p.

Figure 4 shows all the normalized frequencies of lasing and all the threshold gains for the active square microcavity for $\kappa \in [0, 10]$ (we use notations of [35]).

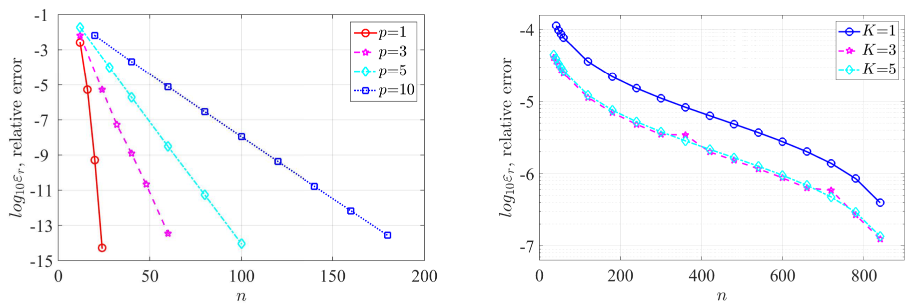

FIG. 2. *Accuracy of calculations for the* $(4, 3, o)$ *mode of the microcavity shaped as the "supercircle" (the left-hand panel) and for the* $(4, 4, ooo)$ *mode of the triangle approximation of the microcavity (the right-hand panel).*

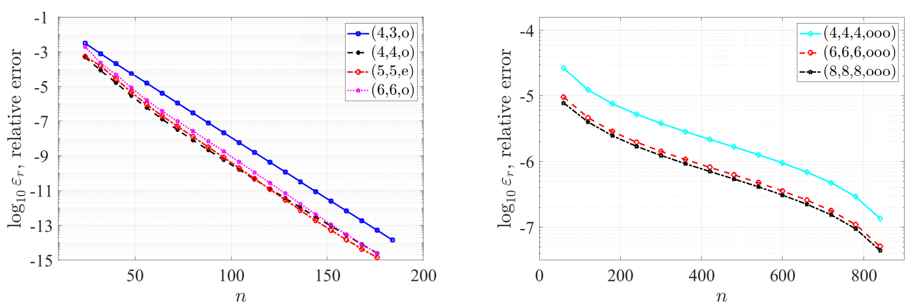

FIG. 3. *Accuracy of calculations for the active square microcavity (the left-hand panel,* $p = 10$*) and for the triangle microcavity (the right-hand panel,* $K = 5$*).*

Figure 5 shows the near-field patterns of the H-polarized modes of the square microcavity. The numerical results shown in Figure 5 agree well with the corresponding results presented in [44] and obtained by the FDTD method and Padé approximation. We find that the qualitative properties of the lasing frequencies and the corresponding fields of the modes (6,6,oooo), (6,6,eeoo), and (8,8,oooo) are in good agreement with characteristics of the modes $(5, 1)$, $(6, 10)$, $(7, 9)$ presented in [44], respectively. We

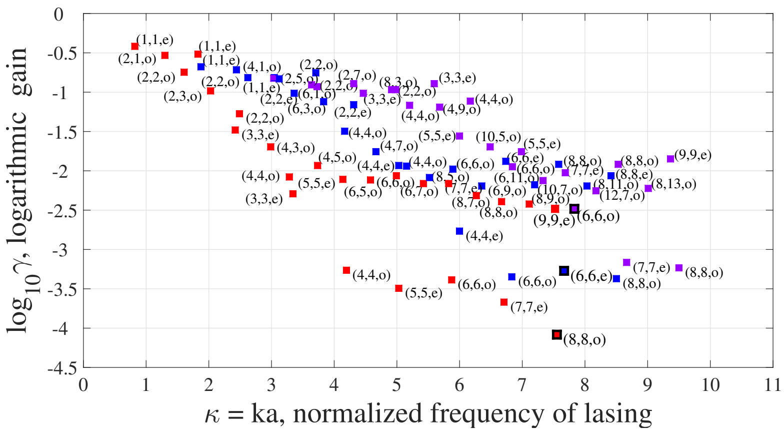

FIG. 4. *Normalized frequencies of lasing and threshold gains for the active square microcavity.*

find also that the maxima of intensity spectrum calculated in [44] correspond to the gains presented in Figure 5.

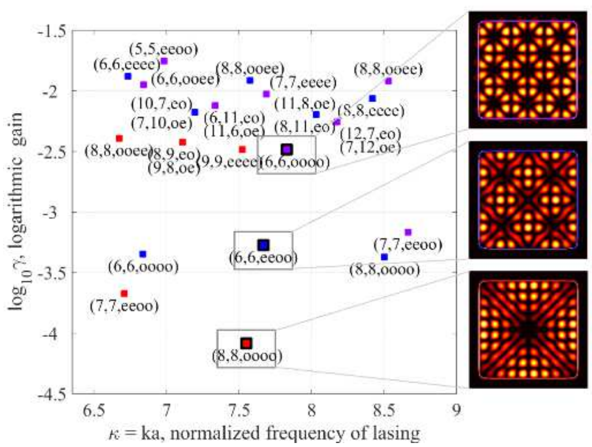

FIG. 5. *Modal fields, normalized frequencies of lasing and threshold gains for the active square microcavity.*

For microcavity shaped as an equilateral triangle with the side a, we use a smooth approximation of Γ characterized by the parametric equation

$$r(t) = \frac{a\sqrt{3}}{4}\left(\cos t - \sum_{k=1}^{K} \alpha_k \cos\left((3k-1)t\right),\ \sin t + \sum_{k=1}^{K} \alpha_k \sin\left((3k-1)t\right)\right), \tag{6.2}$$

where $t \in [0, 2\pi]$, $\alpha_k = \prod_{j=1}^{k} (5-3j)/\left(3^k(3k-1)k!\right)$. Similarly to [3], [4], we compute the function $a/|r''(\pi/3)|$ of the parameter K since, as it was mentioned above, its value should be at least 5–10 times less than $2\pi/(\kappa\alpha_e)$, where $\kappa = ka$. We find that for $\kappa \approx [1, 20]$ it is enough to choose $K \geq 3$. In all our computations we use $K = 5$. Observing Figures 2 and 3 (the right-hand panels), we see that the proposed algorithm converges exponentially, but more slowly than for the square microcavity.

Figure 6 shows the normalized frequencies of lasing and the threshold gains for the triangle microcavity (we use notations [35]).

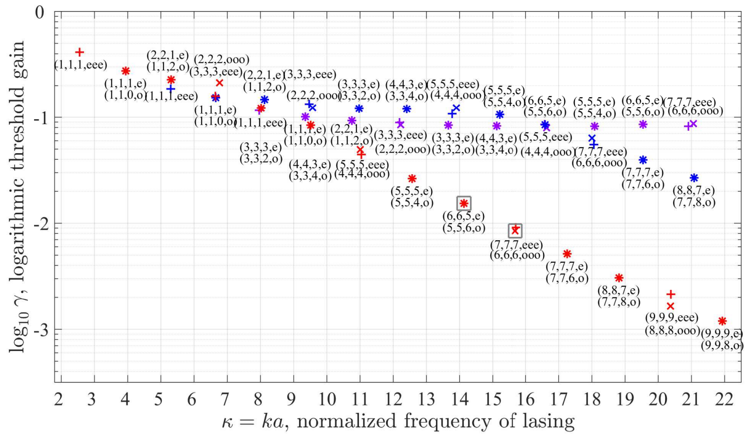

FIG. 6. *Normalized frequencies of lasing and threshold gains for the H-polarized modes of triangle microcavity.*

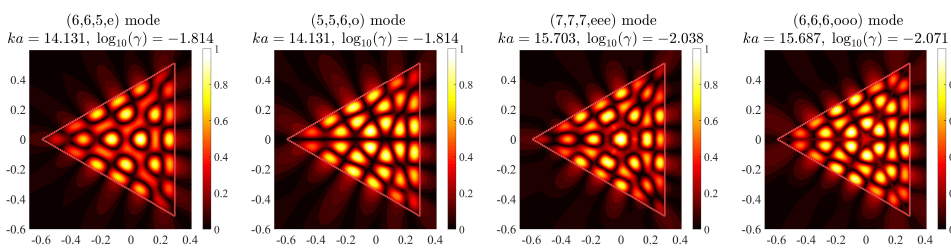

FIG. 7. *Modal fields of the triangle microcavity.*

Figure 7 shows the near-field patterns of the H-polarized modes of the equilateral triangle microcavity. We have compared them with the results of [12] obtained by the FDTD method and Padé approximation for the H-polarized modes in an equilateral triangle optical resonator. We have found that the qualitative properties of the fields are in good agreement.

7. Conclusions. In this paper, we propose and investigate a parametric eigenvalue problem for the Helmholtz equation on the plane based on two physical models of emission from 2-D microcavities: the Complex-Frequency Eigenvalue Problem (CFEP) and the Lasing Eigenvalue Problem (LEP). We call our problem the generalized CFEP (GCFEP). It contains both LEP and CFEP as special cases. Its complex eigenvalues k belonging to the Riemann surface $\mathbb{L}$ of the function $\ln k$ are the possible values of the free-space wavenumber. The corresponding eigenfunctions satisfy the Reichardt radiation condition. The eigenvalues of GCFEP continuously depend on the real-valued loss/gain index, $\gamma \in \mathbb{R}$. If $\gamma \le 0$, that means an open cavity is passive (either lossy or lossless), then the statement of GCFEP corresponds exactly to CFEP. In this case, all the eigenvalues k are located strictly on the lower half of the principal sheet of $\mathbb{L}$, i.e., $\operatorname{Im} k < 0$. If, however, $\gamma > 0$, that is for an active open cavity filled with gain material, then the eigenvalues k are allowed to be located at the real axis of the principal sheet and on its upper half, $\operatorname{Im} k \geq 0$.

As mentioned, the GCFEP eigenfunctions obey the Reichardt radiation condition at infinity in 2-D. If $\operatorname{Im} k = 0$, then this condition is equivalent to the Sommerfeld radiation condition. Hence, if for a positive γ there exists a purely real eigenvalue k, then the pair (k, γ) and the corresponding eigenfunction satisfy all the conditions of LEP. Such a value of γ is the threshold value of the gain index of the material, which fills in the open cavity; it is different for different k.

We reduce GCFEP to a nonlinear eigenvalue problem for the set of Müller boundary integral equations (BIEs) with weakly singular kernels and formulate it as a parametric eigenvalue problem for a holomorphic Fredholm operator-valued function $A(k)$. For each real value of the parameter γ the resolvent set of the operator-valued function $A(k)$ is not empty, its spectrum can only be a set of isolated points on $\mathbb{L}$, which are the eigenvalues of $A(k)$ of finite algebraic multiplicities. For $\gamma \leq 0$, this statement corresponds exactly to CFEP. Each eigenvalue k of the operator-valued function $A(k)$ depends continuously on $\gamma \in \mathbb{R}$. If for some $\gamma > 0$ the intersection of the spectrum of $A(k)$ and the positive real semiaxis of the main sheet of $\mathbb{L}$ is not empty, then it can be only a set of isolated points $k > 0$, which are the eigenvalues of $A(k)$ of finite algebraic multiplicities. This proposition describes the spectrum of LEP.

To solve the obtained parametric nonlinear eigenvalue problem for the set of Müller BIEs numerically we propose the Nyström method and prove the convergence of the numerical scheme. Namely, for any given $\gamma \in \mathbb{R}$, we build a sequence of finite-dimensional holomorphic operator-valued functions $A_n(k)$ that regularly approximates $A(k)$. For every eigenvalue k_0 of $A(k)$, there exists a sequence of the eigenvalues k_n of $A_n(k)$ converging to k_0. If $\{k_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are some sequences of eigenvalues and normalized eigenfunctions of $A(k)$ so that $\{k_n\}_{n\in\mathbb{N}}$ converges to a point k_0 in $\mathbb{L}$, then k_0 is an eigenvalue of $A(k)$, $\{w_n\}_{n\in\mathbb{N}}$ is a discretely compact sequence and its cluster points are normalized eigenfunctions of $A(k_0)$. The estimates of the speed of convergence of $\{k_n\}_{n\in\mathbb{N}}$ to k_0 depend either on the order of the pole k_0 of the operator-valued function $A^{-1}(k)$, or on the algebraic multiplicities of all the eigenvalues of $A_n(k)$ in a neighborhood of k_0, or on the number of the different eigenvalues of $A_n(k)$ in this neighborhood. In each case, the order of convergence with respect to the maximum of the approximation error is equal to or less than one. However, our numerical experiments demonstrate the exponential convergence of the algorithm with respect to the order of the interpolation polynomial.

The approximate solutions coincide well with known exact solutions of LEP and results obtained using other numerical techniques for CFEP. We illustrate our findings with numerical experiments only for $k > 0$ and $\gamma > 0$ since LEP is relatively new and less studied. In forthcoming works, we plan, using the results of the present paper, to mathematically investigate LEP for the cavities having partial active regions and partial lossy regions. A theoretically grounded interpretation on the exponential convergence result supporting the numerical experiments could also be interesting.

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