Optical Theorem Helps Understand Thresholds of Lasing in Open Semiconductor Microcavities

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ABSTRACT
We discuss the application of the Optical Theorem to the analysis of lasing as a linear eigenvalue problem. This yields clear insight into the lasing thresholds of individual modes and easily accounts for the matching of the active-region geometry with the modal electric-field pattern.
Keywords: microcavity lasers, threshold gain, optical theorem, overlap coefficients.

1. INTRODUCTION

Until recently, linear optical modelling of microdisk and other microcavity lasers has implied exclusively the calculation of the natural modes of the *passive* open dielectric resonators [1-3]. Mathematically this means solving the time-harmonic Maxwell eigenvalue problems for the complex-valued natural frequencies, ω, or wavenumbers, $k=\omega/c$. These eigenvalues form a *discrete* set and can be numbered, say, by using the index *s*. However, whilst being meaningful and useful, modelling of lasers via the Q-factors does not address the lasing directly. As discussed in [4], the formulation of the source-free linear time-harmonic electromagnetic problem can be modified by introducing the macroscopic gain, γ, as the *active* imaginary part of the complex-valued refractive index ν assigned to the active region: if the time dependence is assumed as $e^{-i\omega t}$, then $\nu=\alpha-i\gamma$, $\alpha,\gamma>0$. Such a *lasing eigenvalue problem* (LEP) was studied in [5-9]; we call it a "warm-cavity" model to emphasize its position between the cold-cavity passive model and hot-cavity nonlinear one. Note that the gain per unit length, the traditional quantity in the descriptions of the Fabry-Perot cavities, is $g=k\gamma$.

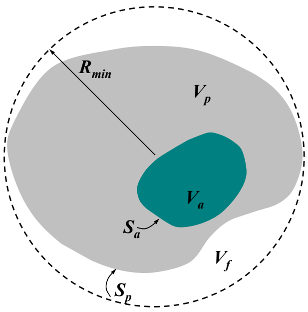

Figure 1. Generic geometry of an open dielectric cavity with a partial active region; minimum sphere determines the cavity volume because outside this sphere the filed contains the outgoing waves only.

Consider a generic laser geometry shown in Fig. 1. Here V_p and V_a are passive and active parts with boundaries S_p and S_a, respectively, and $V_{\min}$ is the so-called minimum sphere, i.e. a sphere of the minimum radius $R_{\min}$ containing both V_p and V_a; note that it may and may not contain a free space part V_f. In LEP, we seek real-valued pairs of numbers, (k,γ), which generate non-zero time-harmonic fields $\{\vec{E},\vec{H}\}$ solving, off the surfaces S_p and S_a, the set of *homogeneous Maxwell equations* with a piecewise-constant refractive index ν equal to α_p in V_p, $\alpha_a-i\gamma$ $(\gamma>0)$ in V_a, and 1 in V_{ext}: rot $\vec{E}=ikZ_0\vec{H}$, rot $\vec{H}=-ik\nu^2 Z_0^{-1}\vec{E}$, where $Z_0=(\mu_0/\varepsilon_0)^{1/2}$ is free-space impedance and all materials are assumed non-magnetic. Across S_p and S_a, the continuity is requested, $\vec{E}^-_{\tan}=\vec{E}^+_{\tan}$, $\vec{H}^-_{\tan}=\vec{H}^+_{\tan}$, where the subscript *tan* is for the field components lying tangential to them. Besides, the time-averaged electromagnetic energy must be locally integrable to eliminate source singularities: $\int_{V\subset\mathbb{R}^3}[\alpha^2 Z_0^{-1}|\vec{E}|^2(\vec{R})+Z_0|\vec{H}|^2(\vec{R})]d\vec{R}<\infty$. Further, a condition at $R\to\infty$ must be added. If domains V_p and V_a are finite and k is real-valued, this is the Silver-Muller condition, $\lim_{R\to\infty}\left\{\vec{E}(\vec{R})-Z_0\vec{H}(\vec{R})\times\vec{R}/R\right\}=0$; it is a vector analogue of the scalar Sommerfeld radiation condition for the

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Helmholtz equation. In the 3-D space, it provides for the spherical-wave behaviour and, in addition, eliminates non-transverse field components at infinity. One can equivalently write (4) as a set of asymptotic requests: $E_\varphi = Z_0 H_\theta \sim \Phi^{(1)}(\theta,\varphi)e^{ikR}/(kR)$, $E_\theta = -Z_0 H_\varphi \sim \Phi^{(2)}(\theta,\varphi)e^{ikR}/(kR)$, and $E_R, H_R \sim 0$, where dimensionless angular functions $\Phi^{(1,2)}(\theta,\varphi)$ indirectly depend on $k,\gamma,\alpha_p,\alpha_a$ and S_p, S_a.

2. OPTICAL THEOREM FOR LASERS

A very instructive insight into the nature of lasing is obtained from the Optical Theorem (OT). As known, in the time-harmonic plane wave scattering OT links the total extinction cross-section of a scatterer with the amplitude of the forward-scattered field in the far zone (e.g. see [10], p. 98). The most general form of such relation is known as the Complex Poynting Theorem; for the complex-valued *k*, this is

$$\Pi = -(1/2)\int_{V_{\min}} (\vec{j}^{e*}\vec{E} + \vec{j}^{m}\vec{H}^{*})dv + (i/2)\int_{V_{\min}} (k^{*}\varepsilon^{*}Z_0^{-1}|\vec{E}|^2 - k\mu Z_0|\vec{H}|^2)dv, \quad \Pi = (1/2)\oint_S \vec{E}\times\vec{H}^{*}ds, \tag{1}$$

where Π is the total outward flux of the Poynting vector averaged over the period of oscillations, through the boundary *S* enclosing $V_{\min}$, $\varepsilon = \nu^2$ and μ are the relative permittivity and permeability, respectively, $\vec{j}^e$ and $\vec{j}^m$ are the given electric and magnetic currents, respectively, and the asterisk means complex conjugation. Now, apply (1) to the *s*-th lasing mode field, $\{\vec{E}_s, \vec{H}_s\}$, taking into account that $\operatorname{Im} k_s = 0$, $\vec{j}^e = \vec{j}^m = 0$, and additionally $\mu = 1$. The result is the OT for lasers:

$$Z_0 \operatorname{Re}\Pi_s(\gamma_s) = \alpha_a \gamma_s k_s \int_{V_a} |\vec{E}_s(\vec{R}, k_s, \gamma_s)|^2 dv \ , \tag{2}$$

which means that the radiation losses are balanced by the gain provided that $\gamma = \gamma_s$. One can show that

$$\Pi \equiv \operatorname{Re}\Pi = 1/(2Z_0k^2)\int_0^{2\pi}\int_0^{\pi}[|\Phi^{(1)}(\theta,\varphi)|^2 + |\Phi^{(2)}(\theta,\varphi)|^2]\cos\theta d\theta d\varphi . \tag{3}$$

The OT for lasers sheds important light on the behaviour of modal thresholds in the cavity with a partial active region. Indeed, introduce for each *s*-th mode the factors $\Gamma_s^{(j)} \le 1$ ($j = a, p, f$) as follows:

$$\Gamma_s^{(a)} = W_s^{(a)}/W_s, \quad W_s^{(j)}(k_s,\gamma_s) = \int_{V_j} \alpha_j^2 |\vec{E}_s(\vec{R},k_s,\gamma_s)|^2 dv, \quad W_s(k_s,\gamma_s) = \int_{V_{\min}} \alpha^2 |\vec{E}_s(\vec{R},k_s,\gamma_s)|^2 dv, \tag{4}$$

where $\alpha = \alpha_{a,p}$ in $V_{a,p}$ and 1 in V_f, and $V_{\min} = V_a + V_p + V_f$. Note that $\Gamma_s^{(a)} + \Gamma_s^{(p)} + \Gamma_s^{(f)} = 1$. This gives way to re-write the OT (2) in the following manner:

$$Z_0\alpha_a \operatorname{Re}\Pi_s(k_s,\gamma_s) = \gamma_s k_s \Gamma_s^{(a)}(k_s,\gamma_s) W_s(k_s,\gamma_s) \ , \tag{5}$$

where the quantity $W_s(k_s,\gamma_s)$ is easily identified as the active-cavity *mode volume* frequently used in the quantum electrodynamics (QED) [11]. $\Gamma_s^{(a)}$ is usually called *mode confinement factor*, although its meaning is rather a measure of the active-region overlap with the modal E-field. Note that in QED these quantities appear from heuristic considerations. In contrast, here we have introduced the modal volume in rigorous and unambiguous way based only on mathematical manipulations with Maxwell equations that can be considered as a formal grounding of definition of both W_s and $\Gamma_s^{(a)}$. Note also that these quantities make sense only as discrete values linked to the modes – this is the point neglected in QED. Further investigation of (5) assuming that the threshold is small, $\gamma_s \ll 1$, shows that asymptotically

$$\gamma_s = Z_0\alpha_a \operatorname{Re}\Pi_s(k_s,0)[k_s\Gamma_s^{(a)}(k_s,0)W_s(k_s,0)]^{-1} + O(\gamma_s^2) \ . \tag{6}$$

Therefore, as a first-order approximation to γ_s, one may take the mode field components as for a passive cavity ($\gamma = 0, k_s = \operatorname{Re} k_s$) and use the formula $\gamma_s = \alpha_a(\Gamma_s^{(a)}Q_s)^{-1}$, where Q_s is the conventional Q-factor.

3. NUMERICAL EXAMPLES

A. Single uniformly active circular cavity

It is interesting to check the OT for a 2-D uniformly active circular cavity of radius *a*, where $V_{\min} = V_a$. In this case (e.g., see [5]), the E_z or Z_0H_z field component of a lasing mode with $m = 0,1,2,...$ is

$$U_{mn}(r,\varphi) = AJ_m(\kappa_{mn}\nu_{mn}r/a)\cos m\varphi, \text{ if } r < a \text{ or } AJ_m(\kappa_{mn}\nu_{mn})[H_m(\kappa_{mn})]^{-1}H_m(\kappa_{mn}r/a)\cos m\varphi, \text{ if } r > a \ , \tag{7}$$

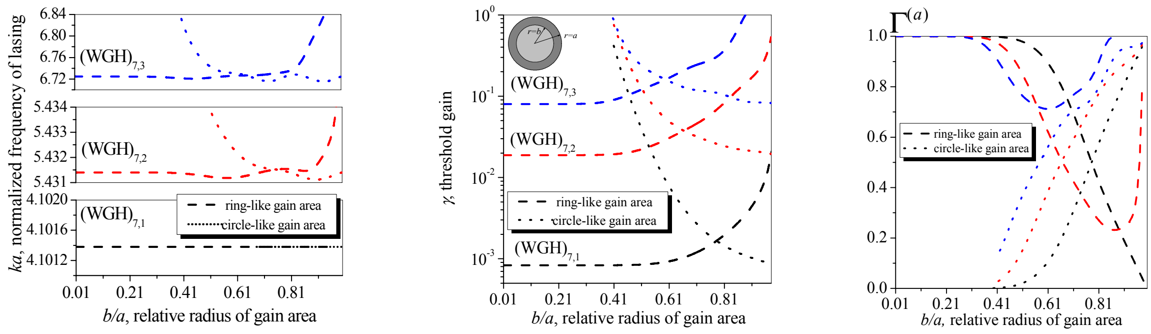
where A is arbitrary constant, $\kappa_{mn}=k_{mn}a$, $\nu_{mn}=\alpha-i\gamma_{mn}$, and $J_m(.)$ and $H_m(.)=H_m^{(1)}(.)$ are the Bessel and Hankel functions, respectively. Here $(\kappa_{mn},\gamma_{mn})$ are the roots of the characteristic equations for the E_z/H_z modes, which are $J_m(\kappa\nu)H_m^{'}(\kappa)-\nu^{\pm1}J_m^{'}(\kappa\nu)H_m(\kappa)=0$. Thus, the OT identity (2) is reduced to

$$\alpha_a\,\mathrm{Im}[J_{m-1}(\kappa_{mn}\nu_{mn})J_m(\kappa_{mn}\nu_{mn}{*})]-\gamma\,\mathrm{Re}[J_{m-1}(\kappa_{mn}\nu_{mn})J_m(\kappa_{mn}\nu_{mn}{*})]=2\alpha_a(\pi\kappa_{mn})^{-1}\,|\,J_m(\kappa_{mn}\nu_{mn})/H_m(\kappa_{mn})\,|^2, \quad (8)$$

Computations show that this identity is satisfied with the same accuracy as the accuracy of finding $(\kappa_{mn},\gamma_{mn})$.

B. Single circular cavity with a radially non-uniform active region

WG modes in a thin disk with a radially piece-wise active region ($\alpha_a=\alpha_p=\alpha$) were studied in [6] within the 2-D LEP. In the uniformly active disk their thresholds behave asymptotically as $\gamma_{mn}=O(e^{4n-4\alpha k_{mn}a/\pi})$, while $k_{mn}a=O(m+2n)$, i.e. are as small as $\gamma_{mn}\approx 0.1$ or less if n=1,2,3 and $m/\alpha<k_{mn}a<m$. The plots in Figs. 2-a and 2-b demonstrate the dynamics of the modal thresholds and frequencies for the WG-modes $(H_z)_{10,n}$ ($n=1,2,3$) in the circular cavity with the active region being either a centred circle of varying radius b or a ring of varying inner radius b. As visible, for each mode the threshold curves cross each other.



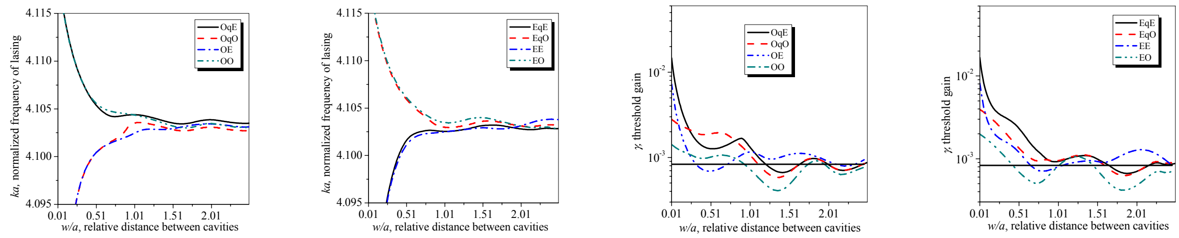
(a) (b) (c)

Figure 2. Normalized lasing frequencies (a), thresholds (b), and active-region mode overlap coefficients (c) of the $(H_z)_{7,n}$ modes (n=1,2,3) in the partially active disk versus the normalized outer (dotted curves) or inner (dashed curves) radius of the concentric circular or ring-like active region, respectively. $\alpha_a=\alpha_p=2.63$.

What is remarkable, at the crossing point threshold is exactly twice higher than in the uniformly active disk (by 0.3 in the logarithmic scale). As the contrast between V_a and V_p is very small, $O(\gamma)$, and the WG-mode frequency and field pattern vary as $O(\gamma)$, expression (6) tells that this can be explained by the drop in the modal gain overlap factor to $\Gamma_s^{(a)}=1/2$. The plots in Fig. 2-c provide evidence to this suggestion. This effect explains the lowering of the threshold pump power reported in [12].

C. Active circular cavity optically coupled with similar passive cavity

A configuration corresponding to the case B in Fig. 1 is the two-disk PM laser with one active and one passive cavity of the same or different size and material, both located in the same plane. In 2-D approximation (Fig. 3-c), the active and passive regions, V_a and V_p, are respectively the circular domains $(r_{1,2}<a_{1,2})$, and the total cavity domain $V_{\min}$ is the circle of the radius $r_{\min}=a_1+a_2+w/2$. Note that all supermodes here split into two orthogonal families, even and odd with respect to the line of symmetry, which is the x-axis, even if the disks radii and materials are the same. This is because the symmetry with respect to the y-axis remains broken by the presence of gain in disk #1 only. In Fig. 3, we show the lasing frequencies and thresholds for the supermodes built on the WG-modes $(H_z)_{7,1}$ in each of two nearly identical (apart of the gain) disks, i.e. for $\alpha_a=\alpha_p$ and $a_1=a_2$. Optical coupling leads to appearance of a quartet of such supermodes grouped in pairs; in each pair, both supermodes have the same parity across the x-axis and, if $a_1=a_2$, remain coupled only due to the presence of gain, γ, in one disk, otherwise they would be independent – see [7]. Resolving these modes when the air-gap increases is computationally difficult task and therefore we present the plots for two disks with slightly different (by $10^{-4}a$) radii. It is interesting to compare such a laser with one isolated active disk [5] and two active disks [7]. One can see that if the coupling is strong ($w/a<0.75$), the pumping of only one disk reduces the threshold exactly in two times (by 0.3 in the logarithmic scale), i.e. $\Gamma_s^{(a)}\approx 1/2$.



(a) (b) (c) (d)

Figure 3. Normalized lasing frequencies (a), (b) and thresholds (c), (d) of the $(H_z)_{7,1}$ supermodes in the coupled active and passive circular cavities versus the normalized rim-to-rim distance. $\alpha_a = \alpha_p = 2.63$.

In contrast, if the separation between the disks is very large the coupling is weak and the passive disk is "turned off", hence $\Gamma_s^{(a)} \approx 1$ and the threshold is close to the one-disk value (straight lines in Fig. 3). Note that due to the optical coupling the threshold may become lower than the one-disk value if the air-gap is selected properly – similarly to two active disks [7].

4. CONCLUSIONS

We have presented the OT expressions for the linear lasing modes that can be used for the verification of numerical results and lead to a simple asymptotic link between the passive-cavity Q-factor and the active-cavity threshold and gain overlap coefficient. Numerical examples demonstrating nontrivial interplay between passive and active parts of the laser cavities in their competition for the mode field have been presented.

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