Effective Mode Volume of a Natural Mode of an Open Dielectric Resonator with an Active Region

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Abstract: We study the lasing eigenvalue problem for a generic open dielectric resonator with gain material. The gain is introduced within the active region via the "active" imaginary part of the refractive index. Each eigenvalue is constituted of two positive numbers, namely, the lasing wavenumber and the threshold value of material gain. This approach yields clear insight into the lasing thresholds of individual modes. The Optical Theorem, if applied to the lasing-mode field, puts familiar "gain=loss" condition on firm footing. It enables us to rigorously introduce the conception of the volume of an open resonator and then the effective-mode volume, both for the passive cavities and cavities with active regions.

The semiconductor, polymeric and crystalline microcavity lasers are among the most promising sources of waves from THz to UV. Their frequencies of lasing are determined mainly by the gain material system used and also the shape and size of the resonator, which is frequently a disk. Performance of such lasers as electron devices critically depends on the proper choice of the current-injection electrodes. As the density of carriers is the largest near the electrodes, their configuration, in fact, determines the location of the active region in the resonator, which can be viewed as dielectric cavity. Design and optimization of such devices relies heavily on the availability of computationally efficient and simultaneously accurate electromagnetic models. Here, two points are important. First, the typical size of the resonators is in the range of fractions to tens of the wavelength that makes the Geometrical Optics based modeling tools impractical. Therefore the use of the Maxwell equations is mandatory. However, FDTD codes popular today fail to characterize the lasing modes directly and suffer of a number of deficiencies. Second, account of the presence of active region is crucial. In the computations, it is necessary to have a simple and practical criterion for validation of numerical results. Our initial goal was deriving such a criterion from the Maxwell equations however obtained results have more general sense.

Consider a generic 3-D open dielectric resonator shown in Fig.1. Here V_d and V_a are passive-dielectric and active-dielectric regions with boundaries S_d and S_a, respectively, and $V_{\min}$ is the so-called minimum sphere, i.e. a sphere of the minimum radius $R_{\min}$ containing both V_d and V_a; note that it may and may not contain a free space part, V_f, hence the whole passive part of the open cavity is $V_p = V_d + V_f$. Importance of the minimum sphere is in the fact that outside of it the field is superposition of solely outgoing waves while inside it also contains the incoming waves.

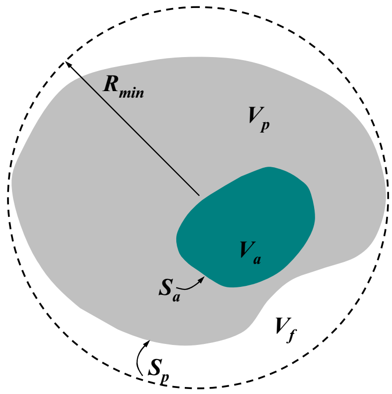

Fig. 1. Open dielectric resonator with a partial active region.

Following [1]-[5], we will be interested in the study of self-excitation threshold conditions of such a resonator. This means that we look for eigensolutions of electromagnetic-field problem characterized by the real-valued pairs of numbers, (k,γ). The first of them is normalized frequency $k=\omega/c$ and the second is material gain. They are the eigenvalues that generate non-zero time-harmonic modal fields $\{\vec{E},\vec{H}\}e^{-i\omega t}$ solving, off S_d and S_a, the homogeneous Maxwell equations with piecewise-constant refractive index ν equal to 1 in V_f and out of $V_{\min}$, α_d in V_d ($\operatorname{Im}\alpha_d \geq 0$), and $\alpha_a - i\gamma$ ($\alpha_a,\gamma>0$) in V_a (materials are non-magnetic). On S_d and S_a, the continuity of the tangential components is requested. Besides, the field energy must be locally integrable to prevent source-like singularities.

Further, a *condition at infinity*, at $R\to\infty$, must be added. If the domains V_d and V_a are finite and k is real-valued, this is

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the Silver-Muller condition of radiation [6].

The fundamental properties of the natural modes (eigenmodes) can be established for an arbitrary open cavity with an arbitrary active region. This is based on the analytical regularisation (see [1],[2]), i.e. equivalent reduction of the eigenvalue problem to a set of the Fredholm second-kind boundary IEs of Muller's type, and the use of the operator extensions of the Fredholm theorems. It is found that the eigenvalues form a discrete set on the plane (k,γ), so that they can be counted with the aid of some index, say *s*; each (k_s,γ_s) has finite multiplicity and depends on S_d, S_a and α_d, α_a in piece-continuous or piece-analytic manner, and this property can be lost only if eigenvalues coalesce. Note also that the gain per unit length, the traditional quantity in the descriptions of the Fabry-Perot cavities, is $g = k\gamma$.

A very instructive insight into the properties of natural modes can be obtained from the Complex Poynting Theorem (CPT) ([6], p. 98). The most general form of such expression, for the complex *k*, is

$$\Pi = -(1/2)\int_V (\vec{j}^{e*}\vec{E} + \vec{j}^{m}\vec{H}^{*})dv \\ +(i/2)\int_V (k^{*}\varepsilon^{*}Z_0^{-1}|\vec{E}|^2 - k\mu Z_0|\vec{H}|^2)dv, \quad (2)$$

where

$$\Pi = (1/2)\oint_S \vec{E}\times\vec{H}^{*}ds \quad (3)$$

is the total outward flux of the Poynting vector through the arbitrary boundary *S* enclosing a volume *V* containing all scatterers and sources, Z_0 is free-space impedance, $\varepsilon = \nu^2$ and μ are the relative permittivity and permeability, respectively, $\vec{j}^e$ and $\vec{j}^m$ are given electric and magnetic currents, respectively, and the asterisk means complex conjugation.

CPT (2) can be also applied to a *natural mode* number *s* (in this case, $\vec{j}^e = \vec{j}^m = 0$). At first, consider a *passive open cavity* (i.e., $V_a = 0$), having complex eigenfrequency k_s. On the extraction of the real part, we retrieve the formula,

$$-\frac{\operatorname{Re}k_s}{\operatorname{Im}k_s} = \frac{W_s}{W_{abs(s)} + W_{rad(s)}}, \quad (4)$$

$$W_s = (1/2)\int_{V_{\min}} (Z_0^{-1}\operatorname{Re}\varepsilon|\vec{E}_s|^2 + Z_0\operatorname{Re}\mu|\vec{H}_s|^2)dv, \quad (5)$$

$$W_{abs(s)} = (1/2)\int_{V_{\min}} (Z_0^{-1}\operatorname{Im}\varepsilon|\vec{E}_s|^2 + Z_0\operatorname{Im}\mu|\vec{H}_s|^2)dv, \quad (6)$$

$$W_{rad(s)} = \operatorname{Re}\Pi_s / \operatorname{Re}k_s, \quad (7)$$

where W_s, $W_{abs(s)}$ and $W_{rad(s)}$ are the powers stored in, absorbed in, and radiated from open cavity, and $\operatorname{Re}\Pi_s$ is the flux of the mode Poynting vector out of the minimum sphere. Either side of (4) is simply twice the quality factor, $2Q_s$ and can be considered as its rigorous definition. It may take only discrete values linked to the mode.

Now, turn to a resonator with active region shown in Fig. 1 and apply (2) to the *lasing mode* field, $\{\vec{E}_s, \vec{H}_s\}$, taking into account that sources are absent, $\operatorname{Im}k_s = 0$, and $V_a \neq 0$. The result is the fundamental *"gain=loss"* expression whose simplified derivation is met in semi-classical theories [7],

$$\tilde{W}_{rad(s)} + \tilde{W}_{abs(s)} = \tilde{W}_{gain(s)} \quad (8)$$

$$\tilde{W}_{gain(s)} = Z_0^{-1}\gamma_s\alpha_a\int_{V_a}|\vec{E}_s(\vec{R},k_s,\gamma_s)|^2 dv, \quad (9)$$

$$\tilde{W}_{abs(s)} = Z_0^{-1}\operatorname{Im}\alpha_d\operatorname{Re}\alpha_d\int_{V_d}|\vec{E}_s(\vec{R},k_s,\gamma_s)|^2 dv, \quad (10)$$

where we use the mark ~ to emphasize that the corresponding quantities are built on the eigensolutions and depend on γ.

Hence, for the *s*-th mode having the wavenumber k_s, the power lost for radiation is balanced by the "negative absorption" (i.e., modal gain as the power generated in the active region), provided that the material gain equals γ_s. So, this is the "gain = loss" condition derived in rigorous way.

However, besides of the real part, CPT expressed as (2) has also the imaginary part that leads to

$$2\operatorname{Im}\Pi_s = k_s\int_V [Z_0^{-1}\operatorname{Re}(\nu^2)|E_s|^2 - Z_0|\vec{H}_s|^2]dv \quad (11)$$

where the domain *V* is arbitrary. In the limit of $S \to \infty$ as a circle of large radius, the left-hand part of (11) is zero due to the radiation condition, and the same is valid if $V = V_{\min}$ because of the continuity of Π. Therefore, we obtain that

$$\int_{V_{\min}} [Z_0^{-1}\operatorname{Re}(\nu^2)|\vec{E}_s|^2 - Z_0|\vec{H}_s|^2]dv = 0, \quad (12)$$

which means that the fractions of the power contained in the electric and magnetic field of any mode inside the cavity volume $V_{\min}$ equal each other. The same is valid in the whole space. Note that this property holds true for any mode in both passive and active open cavities, on resonance.

The laser configurations where the active region does not coincide with the whole cavity (i.e. $V_p \neq 0$ in Fig. 1) are frequently met in practice. For instance, it is realized if one uses a sharply focused pump beam in optically pumped laser or, alternatively, the pump beam goes through an axicon. Besides, combination of separated active and passive regions is typical for the cavities with distributed Bragg reflectors and for the coupled-microcavity lasers using selective pumping. Moreover, this situation is common for all injection lasers, which are known as extremely vulnerable to the proper placing of electrodes.

The CPT for lasers sheds important light on the behavior of modal thresholds in the cavities with partial active regions. Indeed, can introduce the quantity $\Gamma_s^{(a)} \le 1$ given by

$$\Gamma_s^{(a)} = \tilde{W}_s^{(a)} / \tilde{W}_s,$$
$$\tilde{W}_s^{(a)}(k_s, \gamma_s) = (1/2Z_0)(\alpha_a^2 - \gamma_s^2)\int_{V_a} |\vec{E}_s(\vec{R}, k_s, \gamma_s)|^2 dv, \quad (13)$$
$$\tilde{W}_s(k_s, \gamma_s) = (1/2Z_0)\int_{V_{\min}} \mathrm{Re}(\nu^2)\, |\vec{E}_s(\vec{R}, k_s, \gamma_s)|^2 dv,$$

where $\nu = \alpha_d$ in V_d and 1 in V_f, and $V_{\min} = V_a + V_d + V_f$.

From this definition it is clear that $\Gamma_s^{(a)}$ is the fraction of E-field power contained in the active region. It is also the *overlap coefficient* between the active region and the modal *E*-field (a.k.a. *mode confinement factor*). This is a strictly discrete quantity having values linked to specific modes.

This enables us to re-write CPT (8) as follows:

$$\gamma_s = \alpha_a [\Gamma_s^{(a)}(k_s, \gamma_s)\ \tilde{Q}_s(k_s, \gamma_s)]^{-1}, \quad (14)$$

where now

$$\tilde{Q}_s = \tilde{W}_s / [\tilde{W}_{rad(s)} + \tilde{W}_{abs(s)}] \quad (15)$$

is the Q-factor of the *active cavity.* Further investigation of (14) assuming that the threshold is small, $\gamma_s \ll 1$, shows that the first-order approximation to γ_s is obtained if one takes the mode field components and the frequency as for a passive cavity ($\gamma_s = 0$),

$$\gamma_s = \alpha_s [\Gamma_s^{(a)}(k_s, 0) Q_s]^{-1} + O(\gamma_s^2) \quad (16)$$

Expression (16) tells that in order to achieve low threshold in the active (pump on) cavity, it is not enough to have high Q-factor of the same mode in the passive (pump off) cavity. The mode E-field overlap with active region is equally important and can dramatically counterbalance the Q-factor – this happens, for instance, with the quasi-WG modes in a stadium-cavity laser if the electrode is placed in the cavity centre.

If the modal electric field value in (5) or (13) is normalized by its maximum, then W_s or $\tilde{W}_s$ is the *effective mode volume* for a passive or an active cavity, respectively. Here, in view of (12) it is enough to take account of the first term of (5).

Effective mode volume plays very important role in the cavity quantum electrodynamics (QED) [8]. However, in cavity QED this quantity appears from heuristic considerations; besides, one and the same definition is used for active and passive cavities; the integration is usually taken only over V_p but sometimes is extended to a part of space outside the dielectrics and even outside of $V_{\min}$. In contrast, here we have introduced $\tilde{W}_s$ in rigorous and unambiguous way based only on mathematical manipulations with Maxwell equations, i.e. from first principles.

Note that in the cavity QED it is assumed that the smaller the effective mode volume, the lower the threshold of lasing. Our formula (14) convincingly shows that from the viewpoint of Maxwell equations this is not true. In fact, the role of the effective mode volume is just opposite as $\tilde{W}_s$ enters the denominator of r.h.p. of (14); however this role is balanced by the mode emission loss, $\tilde{W}_{rad(s)}$ in the numerator - these two quantities “breath” together. The real figure-of-merit of the mode in active cavity is its Q-factor, $\tilde{Q}_s$, which can be approximated by the passive-cavity counterpart, Q_s.

Thus, the Optical Theorem for the lasers considered in linear formulation has enabled us to propose a rigorous mathematical definitions of the open-resonator volume (as a minimum sphere), the effective mode volume, and the mode-active-region overlap coefficient – this, in fact, provides grounding to these quantities, widely used in semi-classical laser physics and QED as phenomenological ones.

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