Radiation conditions for open waveguides

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It is known that in the case of harmonic oscillations ($\sim e^{-ikct}$, $\mathrm{Im}\,k=0$), the radiation condition is one of the principles for deriving a unique solution of Maxwell's equations in an infinite domain.[1-4] If the boundaries extend to infinity, then Sommerfeld's radiation condition[1] is inapplicable. For a hollow closed waveguide, the radiation condition was derived by Sveshnikov.[5] For open waveguides, on the other hand, similar conditions have not yet been formulated in explicit form, although in a series of works the solution derived from physical considerations is in agreement with the requirement for the "absence of incoming waves" and has yielded reliable results. The aim of the present work is the correct formulation of the radiation condition for an open waveguide, which extends the Sommerfeld and Sveshnikov's conditions to the case of unbounded space containing bodies and surfaces that are infinite and regular along an axis (Fig. 1).

<u>Definition 1.</u> We shall examine a regular open waveguide of compact cross section which is formed by a finite number of continuous nonintersecting elements of three types: a) a dielectric cylinder of cross section D with a finite boundary ∂D; b) an ideally conducting cylinder of cross section M with a finite boundary ∂M_1; and c) ideally conducting infinitely thin open surfaces of cross section ∂M_2. We shall designate $\partial M=\partial M_1\cup\partial M_2$, $\bar M = M\cup\partial M$, $W=\bar M\cup\partial D$. We shall call the open waveguide described above the waveguide W.

All components of the fundamental solution of Maxwell's equations $\{E^G, H^G\}$ for the open waveguide W are uniquely determined in terms of two fundamental functions $G^{e,m}(R, R_0)=\{U^{e,m}, V^{e,m}\}$ the solutions of the following boundary-value problem.

<u>Problem (G):</u>

$$\bar G^{e,m}=\left(\frac{\partial^2}{\partial z^2}+k^2\epsilon\right)\bar\Pi^{e,m}\quad(z\neq z_0);\quad \bar\Pi^{e,m}=\{\Pi_1^{e,m},\Pi_2^{e,m}\},$$
$$(\Delta+k^2\epsilon)\,\bar\Pi^{e,m}(R,R_0)=-\bar D^{e,m}(R,R_0);\quad R,R_0\in\mathcal{R}^3\setminus(W\times z), \tag{1}$$

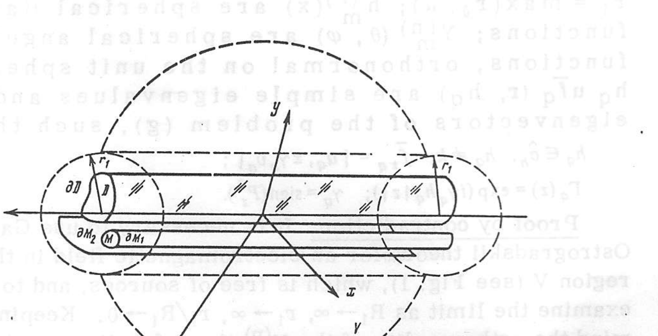

FIG. 1

$$E_\tau^G|_{\partial M\times z}=0;\quad [E_\tau^G]|_{\partial D\times z}=0;\quad [H_\tau^G]|_{\partial D\times z}=0, \tag{2}$$
$$\int_V (|k\bar G|^2+|\mathrm{grad}\,\bar G|^2)\,dv<\infty,\quad V\subset\mathcal{R}^3 \tag{3}$$

where E_τ^G, H_τ^G are the tangential components of the field, $\bar D^e=\{\epsilon^{-1}\delta(R-R_0), 0\}$, $D^m=\{0, \delta(R-R_0)\}$.

Problem (G) is the problem of the diffraction of an intrinsic field of a longitudinal electric or magnetic dipole by the elements of the open waveguide W. To solve this problem, we shall apply an integral Fourier transform

$$\bar G^{e,m}(R,R_0)=(2\pi)^{-1}\int_{-\infty}^{\infty}\tilde\kappa^2(r)\,\bar g^{e,m}(r,r_0;h)\,e^{ih(z-z_0)}\,dh, \tag{4}$$
$$\tilde\kappa^2(r)=k^2\epsilon(r)-h^2,$$

where the integral is understood in the general sense, since the functions $\bar g^{e,m}(r, r_0; h)=\{u^{e,m}, v^{e,m}\}$ can have singularities on the contour of integration.

The functions $\bar g^{e,m}(r, r_0; h)$ are called transverse Green's functions, which are solutions of the following boundary-value problem.

<u>Problem (g):</u>

$$[\Delta+\tilde\kappa^2(r)]\,\bar g^{e,m}(r,r_0;h)=-\bar d^{e,m}(r,r_0);\quad r,r_0\in\mathcal{R}^2\setminus W, \tag{5}$$

where $\mathrm{Im}\,h=0$, $\bar d^e=\{\epsilon^{-1}\delta(r-r_0), 0\}$, and $\bar d^m=\{0, \delta(r-r_0)\}$, with conditions of the type in Eqs. (2) and (3) in the plane of the cross section. In addition, by virtue of the compactness of $W\subset\mathcal{R}^2$ in the limit $r\to\infty$, the functions $\bar g^{e,m}(r, r_0; h)$ necessarily are subject either to the condition of exponential decay (for $|h|>k$) or to the Sommerfeld condition (for $|h|<k$). Following Sveshnikov[6] (see also Refs. 7 and 8), we shall write them in the form (we shall drop the indices e and m since they are unimportant)

$$\bar g(r,r_0;h)=\sum_{n=-\infty}^{\infty}\bar a_n H_n^{(1)}(\kappa r)\,e^{in\varphi}\quad(r\to\infty), \tag{6}$$

where $\kappa=k^2-h^2$; $\mathrm{Im}\,h=0$; $\mathrm{Re}\,\kappa\ge 0$, $\mathrm{Im}\,\kappa\ge 0$; and $\bar a_n=\{a_n^u, a_n^u\}$.

The problem of determining $\bar G(R, R_0)$ in $\mathcal{R}^3$ by way of calculating the integrals in Eq. (4) leads to the need to study $\bar g(h)$ as a function of the parameter h in the range of its analytic continuation. It is easy to see that this region forms a subset of the Riemann surface $\mathcal{K}$ of the function $\mathrm{Ln}\,\kappa(h)$, which is the region of analyticity of the fundamental solution of Eq. (5) in the absence of the waveguide W. We shall designate $\mathcal{K}_0$ to be that ("physical") sheet of $\mathcal{K}$, over the real axis of which Eq. (6) holds.

Applying the Gauss–Ostrogradskii theorem, known in electrodynamics as the complex power theorem, to the field in a cylinder of radius $r_1>a$, where a is the radius of the minimum circle surrounding the waveguide, we come to the following assertion.

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Lemma 1. There do not exist any eigenvalues of the problem (g) for which $h \in \mathcal{H}_0$, $\operatorname{Im} h = 0$ and

$$4(\operatorname{Re} k \operatorname{Re} \kappa + \operatorname{Im} k \operatorname{Im} \kappa) e^{-2 \operatorname{Im} \kappa r_1} \sum_{n=-\infty}^{\infty} (|a_n^u|^2 + |a_n^v|^2) \neq -\int_{S_{r_1}} [\operatorname{Im} k H^2 + (\operatorname{Re} k \operatorname{Im} \epsilon + \operatorname{Im} k \operatorname{Re} \epsilon) E^2]\, ds \tag{7}$$

and (or)

$$4(\operatorname{Re} k \operatorname{Im} \kappa - \operatorname{Im} k \operatorname{Re} \kappa) e^{-2 \operatorname{Im} \kappa r_1} \sum_{n=-\infty}^{\infty} (|a_n^u|^2 - |a_n^v|^2) \neq -\int_{S_{r_1}} [(\operatorname{Re} k \operatorname{Re} \epsilon - \operatorname{Im} k \operatorname{Im} \epsilon) E^2 - \operatorname{Re} k H^2]\, ds. \tag{8}$$

From Lemma 1 and from the fact that for $\epsilon = 1$ the problem (g) decomposes into two problems for u and v, which are independent, relative to the scalar functions we find

Theorem 1. The boundary-value problem (g) has a unique solution in the following cases: a) for all real h, if $\operatorname{Im} k > 0$, $\operatorname{Im} \epsilon = 0$; b) for all real $h \neq \pm k$, if $\operatorname{Im} k = 0$ and if either $\operatorname{Im} \epsilon > 0$, or $\epsilon = 1$ ($h = \pm k$ are branch points); c) for all real $|h| < k$, if $\operatorname{Im} k = 0$, $\operatorname{Im} \epsilon = 0$, $\operatorname{Re} \epsilon \neq 1$.

The following fundamental result holds.

Theorem 2. The transverse Green's functions $\bar{g}^{e,m}$ (r, r_0; h) of the open waveguide W exist, are analytic over h on the segment $|h| < k$, $h \in \mathcal{H}_0$, and permit analytic continuation on the Riemann surface $\mathcal{H}$, with the exception of a set σ_h (which is no more than denumerable and symmetric relative to 0) of isolated points, which are poles of finite multiplicity, without finite cluster points.

For an open waveguide, comprised of elements of type b) and c) from definition 1, the proof of Theorem 2 is a corollary of the result of Refs. 9 and 10. For elements of type a), the same result can be obtained by investigating integral equations that are equivalent to problem (g) on the basis of their Fredholm nature and analytic dependence of the kernels on h.[12]

Following Refs. 7 and 13, it can be established that the residues of the transverse Green's function at the points $h \in \sigma_h$ are equal, within a constant, to the eigenfunctions $\bar{f}_q(r, h_q)$ of the homogeneous problem (g) with a generalized radiation condition in the limit $r \to \infty$, which formally coincides with (6) for simple poles, with one difference that $h_q \in \mathcal{H}$. It is easy to see that in this case (6) allows an exponential increase as $r \to \infty$. The functions $\{E_q, H_q\}$, which are uniquely determined by the eigenvectors $f_q(r, h_q)$, will then describe the fields of the generalized natural waves of the open waveguide W, which correspond to a discrete spectrum σ_h of the propagation constants h.

Definition 2. We shall designate $\hat{\sigma}_h = \bigcup_{q=1}^{Q} h_{\pm q} : \{h_{-q} = -h_q;\ h_q \in \sigma_h;\ h_q \in \mathcal{H}_0;\ \operatorname{Im} h_q = 0\}$ to be the set of propagation constants of all proper natural waves of the open waveguide W.

It is obvious that in the limit $r \to \infty$ $\bar{f}_q(r, h_q) = \bar{O}[\exp^{-1}(h_q^2 - k^2)^{1/2} r]$, such that there exists the finite number

$$P_z^q = c(8\pi)^{-1} \operatorname{Re} \int_{\mathbb{R}^2 \setminus W} [E_q \times H_q^*]\, z_0\, ds, \tag{9}$$

i.e., the average flux of complex power through the cross section of the open waveguide.

Theorem 3. The principle of ~~finite~~ limiting absorption for the derivation of a unique solution for the problem (g) is correctly applied only if either $\epsilon = 1$, $h \neq \pm k$, or $\operatorname{Im} \epsilon > 0$, $h \neq \pm k$, or $|h| < k$, or k does not belong to $\hat{\sigma}_h$. Otherwise, the limit of $\bar{g}(r, r_0; h, k)$ for $\operatorname{Im} k \to +0$ exists only in a class of generalized functions.

The proof of Theorem 3 is derived from Theorems 1 and 2.

Starting from Theorem 2, we shall now construct only that fundamental Green's function $\bar{G}(R, R_0)$, which will b in agreement with the requirement for the absence of sources at infinity, and we shall prove its uniqueness. Studying its asymptotic behavior as $R \to \infty$ permits one to formulate the radiation condition for an open waveguide.

For simplicity, we assume that all points $\hat{\sigma}_h \neq \phi$ are simple and do not coincide with $\pm k$. We apply the method of steepest descents to the study of the integral in Eq. (4). The following lemma, following from the results of Refs. 13 and 15, is also necessary.

Lemma 2. If $h_p, h_q \in \hat{\sigma}_h, p \neq q$, then the following orthogonality relations hold:

$$\int_{\mathbb{R}^2 \setminus W} [E_q \times H_p^*]\, z_0\, ds = 0; \qquad \int_{\mathbb{R}^2 \setminus W} [E_q \times H_p]\, z_0\, ds = 0. \tag{10}$$

Theorem 4. The boundary problem (G) has for $\operatorname{Im} k \geq 0$, $\operatorname{Im} \epsilon \geq 0$ a unique solution in $\mathbb{R}^3$, which satisfies for $R = (r^2 + z^2)^{1/2} \to \infty$ the condition

$$\bar{G}(R, R_0; k) = \begin{cases} \sum\limits_{n=-\infty}^{\infty} \sum\limits_{m=-n}^{n} \bar{b}_{mn} h_m^{(1)}(kR) Y_m^{(n)}(\theta, \varphi), & r > r_1 \\ o(1), & r < r_1 \end{cases} + \sum_{q=1}^{Q} \alpha_q \bar{f}_{\pm q}(r, h_q) \Gamma_q(z) \quad (z \gtrless 0), \tag{11}$$

$r_1 = \max(r_0, a)$; $h_m^{(1)}(x)$ are spherical Hankel functions; $Y_m^{(n)}(\theta, \varphi)$ are spherical angular functions, orthonormal on the unit sphere; h_q u $\bar{f}_q(r, h_q)$ are simple eigenvalues and eigenvectors of the problem (g), such that

$$h_q \in \hat{\sigma}_h,\ h_q \neq k;\ \bar{f}_{\pm q} = \{u_q, \pm\gamma_q v_q\};$$
$$\Gamma_q(z) = \exp(i\gamma_q h_q |z|);\quad \gamma_q = \operatorname{sign}(P_z^q).$$

Proof by contradiction. It is necessary to use Gauss–Ostrogradskii theorem for an electromagnetic field in the region V (see Fig. 1), which is free of sources, and to examine the limit as $R_1 \to \infty$, $r_1 \to \infty$, $r_1/R_1 \to 0$. Keeping in mind the orthogonality of the $Y_m^{(n)}(\theta, \varphi)$ functions and Lemma 2 and studying the result in relation to its fixed sign behavior, we find a contradiction.

Note that if the multiplicity of any pole is higher than 1, this can be easily allowed for by using the calculus of residues. If any of the simple poles $h_p \to k$, then its contribution needs to be evaluated by a modified method of steepest descent.[14]

Thus, it is necessary to add to a spherical wave that satisfies the Sommerfeld condition [first term in Eq. (11)], the superposition of proper intrinsic waves of the open waveguide (second term). For each of the latter a partial radiation condition can be posed for $|z| \to \infty$:

$$\frac{d\Gamma_q(z)}{d|z|} - i\gamma_q h_q \Gamma_q(z) = 0, \quad q = 1, \ldots, Q, \quad h_q \in \hat{\sigma}_h. \tag{12}$$

Condition (12) is similar to the Sveshnikov's condition[5] for a hollow closed waveguide. It is different, however, in that it has the value γ_q which reflects the fact that in the open waveguide the intrinsic waves can have phase and group velocities in different directions. Thus, in an open waveguide it is insufficient to require the absence of incoming waves, rather, it is necessary to require the absence of waves carrying energy from infinity.

An obvious corollary of Theorems 3 and 4 is

Theorem 5. The principle of finite absorption yields a unique solution of the problem (G), satisfying condition (11) in the limit Im k → +0.

Using the same results as those used in proving Theorem 4, we shall establish the validity of the following proposition.

Theorem 6. For any fields $\{E_1, H_1\}$ and $\{E_2, H_2\}$, satisfying Maxwell's equations and condition (11), the following relation is satisfied, regardless of the complexity of the complex number k:

$$\lim_{\substack{R_1 \to \infty;\, r_1 \to \infty \\ r_1/R_1 \to 0}} \oint_S \{[E_1 \times H_2] - [E_2 \times H_1]\}\; n_s\, ds = 0. \tag{13}$$

Theorem 6 permits the construction of a solution for the problem of the excitation of an open waveguide, i.e., problems of the type (G) with a finite right side, in the form of convolutions with a fundamental solution determined by the functions $\overline{G}^{e,m}(R, R_0)$. In addition, Eq. (13) permits applying Green's vector theorem to constructing integral equations in the theory of diffraction of intrinsic waves (traveling from $z = \pm\infty$) by inhomogeneities in an open waveguide of the screen and inclusion type, compact in $\mathbb{R}^3$. Moreover, the fact that Eq. (13) is also satisfied for all complex k, permits one to correctly formulate the problem of the spectrum of intrinsic oscillations of open resonators that are Loaded with open waveguides. Such generalized intrinsic oscillations satisfy the homogeneous boundary-value problem of the type (G) with the condition (11) for Im k < 0. In this case, Eq. (13) guarantees the construction of equivalent integral equations, despite the increase in the solution in the limit $R \to \infty$.

In conclusion, we note that in the space of two measurements of $\mathbb{R}^2$, the analog of condition (11), for $r = (y^2 + z^2)^{1/2} \to \infty$, is

$$G(\mathbf{r}, \mathbf{r}_0; k) = \left\{ \begin{array}{ll} \sum_{n=-\infty}^{\infty} a_n H_n^{(1)}(kr) e^{in\varphi}, & y > y_1 \\ o(1), & y < y_1 \end{array} \right\} + \sum_{q=1}^{Q} \alpha_q u_q(y, h_q) e^{ih_q|z|} \tag{14}$$

Here it is assumed that there are no ~~reflected~~ backward waves in this case, $\gamma_q \equiv 1$. Using Eq. (14), we can generalize radiation condition (11) to an open waveguide with regularly noncompact cross section. To do this, it is necessary to construct a transverse Green's function $\bar{g}(\mathbf{r}, \mathbf{r}_0; h)$, replacing condition (6) by condition (14). In addition, all of the radiation conditions studied above can be generalized as well for a regularly periodic open waveguide.

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