An electromagnetic analog of a Helmholtz resonator

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It is well known in acoustics that for a rigid sphere with a small hole there exists a resonance frequency in the longwave range,[1-3] at which the amplitude of the oscillations inside the cavity and also the total and the back scattering cross sections increase sharply. In this scattering regime the sphere with a hole is called a Helmholtz resonator.[1]

The analog of the Helmholtz resonator can be realized also in electrodynamics, for example, in the form of a metallic cylinder with a longitudinal slit.

In the present communication it is shown by a rigorous method[4,5] that in the diffraction of a plane H-polarized electromagnetic wave at a circular cylinder with a slit for certain values of the frequency in the longwave range the scattering cross sections behave in the same way as in the case of an ordinary Helmholtz resonator.[3]

Let a plane wave be incident on a cylinder of radius a with an infinite longitudinal slit of angular dimension 2θ oriented in the direction φ_0 (Fig. 1). The field scattered by the cylinder, which must satisfy the Helmholtz equation, the Neumann boundary conditions, the radiation condition, and the condition of finiteness of the energy in any volume in space, can be expressed in the form of the potential of a double layer distributed along the surface of the cylinder with current density $\mu(\varphi)$:

$$H_z(r,\varphi) = \frac{i}{4}\int_L \mu(\varphi_s)\frac{\partial}{\partial\nu}H_0^{(1)}(k|p-p_s|)\,dl_s, \tag{1}$$

where $k=\omega/c$, $p=\{r,\varphi\}$, $p_s=\{a,\varphi_s\}\in L$, L is the contour of the transverse section of the cylinder, dl_s is the arc element of the contour L, ν is the normal to the contour, and $H_0^{(1)}(x)$ is the Hankel function of the first kind.

Using the Fourier series expansion of the function $\mu(\varphi)$ and subjecting the field (1) to the boundary conditions we obtain a system of paired summator functional equations of the first kind with the kernel in the form of trigonometric functions involving the Fourier coefficients of the current density μ_m:

$$\sum_{m=-\infty}^{\infty}\mu_m J_m'(ka)H_m^{(1)\prime}(ka)e^{im\varphi}=-\sum_{m=-\infty}^{\infty}i^m J_m'(ka)e^{im\varphi},\quad 0<|\varphi-\varphi_0|<\pi,$$
$$\sum_{m=-\infty}^{\infty}\mu_m e^{im\varphi}=0,\quad |\varphi-\varphi_0|<0. \tag{2}$$

It is well known that functional equations of the first kind of type (2) are unsuitable for analytical or numerical determination of the unknowns μ_m. However, Eqs. (2) can be regularized. For this purpose it is necessary to have a linear operator determined by the left-hand sides of Eqs. (2), to separate it into the principal and completely continuous parts, and then invert the principal part using the rigorous method of the conjugation problem.[4] This results in an infinite system of linear algebraic equations of the second kind, which is equivalent to (2):

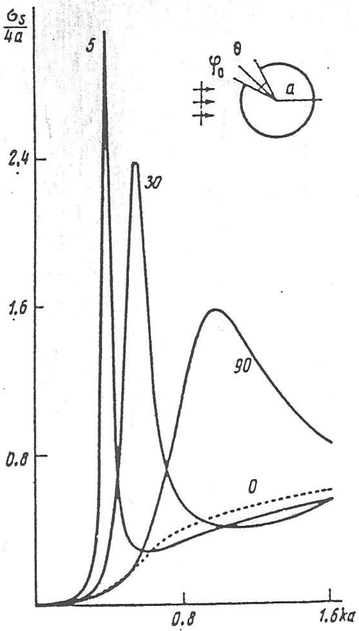

FIG. 1. The total scattering cross section of a cylinder with a slit as a function of the frequency. φ_0 = 180°. The digits on the curves denote the values of θ in degrees.

$$\mu_m=\sum_{n=-\infty}^{\infty}A_{mn}\mu_n+B_m,\quad m=0,\pm1,\ldots, \tag{3}$$

where

$$A_{m0}=i\pi(ka)^2J_0'(ka)H_0^{(1)\prime}(ka)T_{m0},\quad A_{mn}=|n|\delta_nT_{mn};$$

$$B_m=-i\pi(ka)^2\sum_{n=-\infty}^{\infty}i^nJ_n'(ka)T_{mn};$$

$$\delta_n=1+\frac{i\pi(ka)^2}{|n|}J_n'(ka)H_n^{(1)\prime}(ka);$$

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and the functions $W_n(-u)$, $V_{m-1}^{n-1}(-u)$ are defined in Ref. 4; $u = \cos\theta$.

The matrix operator of system (3) is a Fredholm operator[4]; therefore, in the case of arbitrary parameters of the problem the coefficients μ_m can be determined by the method of reduction without any additional justification.

Furthermore, in the region of our interest $ka \ll 1$ system (2) can be solved by the method of successive approximations, which follows from the estimate of the canonical norm of its matrix

$$q = \max_{m\neq 0} \sum_{n\neq 0} |A_{mn}| < C(ka)^2[(1-u)+\sqrt{1-u^2}].$$

Taking $q < 1$, in the zero-order approximation for the Fourier coefficients of $\mu(\varphi)$ we get

$$\mu_m = (-1)^m e^{-im\varphi_0} \frac{(ka)^2}{m} V_{m-1}^{-1}(-u)\mu_0[1+O(ka)],$$
$$\mu_0 = \frac{-i\pi(ka)^2(ka+i\cos\varphi_0 \ln^{-1}\sin{}^1/_2\theta)}{\ln^{-1}\sin{}^1/_2\theta + 2(ka)^2 + i\cdot{}^1/_2\pi(ka)^4}[1+O(ka)]. \quad (4)$$

A computation of the surface current density $\mu(\varphi)$ using the results of analysis of free oscillations in a cylinder with a slit[6] shows that in the region $ka \ll 1$, $|\ln^{-1}\theta| \ll 1$,

$$\mu(\varphi) = \frac{2(ka)^2}{i\pi a}\left\{\mu_0 + i\frac{\pi}{2}ka + O[(ka)^2]\right\} S(\varphi-\varphi_0), \quad (5)$$

where

$$S(\varphi) = \begin{cases} 0, & |\varphi| < \theta, \\ \ln \dfrac{\sin{}^1/_2|\varphi| + \left[\dfrac{(\cos\theta - \cos\varphi)}{2}\right]^{1/2}}{\sin{}^1/_2\theta}, & \theta < |\varphi| < \pi. \end{cases}$$

The current (5), induced by the plane wave in the cylinder with a slit, is small at all frequencies except the resonance frequency

this is a misprint; see denominator of (4) for the correct formula; this journal did not send proofs

$$k_0 a = (-2\ln\sin{}^1/_2\theta)^{-1/2} + O(\ln^{-3/2}\sin{}^1/_2\theta), \quad (6)$$

at which it increases in order of magnitude to $O[(k_0a)^{-1}]$, $k_0a \ll 1$. This results in a sharp increase of the energy characteristics of the scattered field determined by it, i.e. of the total (σ_s) and back (σ_b) scattering cross sections, given by the formulas

$$\sigma_s = \frac{4}{k}\,\mathrm{Re}\sum_{m=-\infty}^{\infty} \mu_m i^m J_m'(ka), \quad \sigma_b = \frac{4}{k}\left|\sum_{m=-\infty}^{\infty} \mu_m(-i)^m J_m'(ka)\right|^2. \quad (7)$$

The frequency dependence of the total scattering cross section computed by the method of reduction from the solution of system (3) is shown in Fig. 1. It has a well-defined resonance character. It follows from (4), (7) that at the resonance frequency (6) we have

$$\sigma_s \simeq \sigma_b = \frac{4}{k_0} + O(k_0 a^2) \quad (8)$$

Thus, on a metallic cylinder with transverse dimensions small compared to the wavelength it is sufficient to cut a slit of the resonance width [in accordance with (6)] in order to obtain a substantial increase in the effective electrodynamic dimensions of the cylinder. For example, for a slit of width $2\theta = 10°$, the energy reflected from the cylinder in the direction to the source at resonance is 30 times greater than from a continuous cylinder of the same dimensions. In other words, whereas the continuous cylinder is practically "invisible" in this wavelength range, a cylinder of the same diameter with a slit is a very significant obstacle for an H-polarized electromagnetic wave.

We note that according to (5), (6) the resonance frequency has a strong dependence on the width of the slit θ and tends to zero for $\theta \to 0$, but has a weak dependence on the angle of orientation of the slit φ_0. The maximum values of the scattering cross section at resonance (8) also have a weak dependence on φ_0. (This dependence is contained in the terms proportional to $O[(k_0a)^2]$.) These characteristics are obviously accounted for by the long-wave nature of the phenomenon in question. For the same reason the scattering of the incident field is almost isotropic. As regards the field in the near zone and inside the cylinder, at the resonance it has the form shown in Fig. 2. The electric field (parallel to the lines H_z = const) is concentrated at the slit, and the magnetic field is maximum inside the cylinder, in the same way as in the acoustic Helmholtz resonator the kinetic energy of the acoustic oscillations is maximum at the aperture and the potential energy of compression is maximum in the cavity. In fact it can be shown that in the present case the single nonzero component of the magnetic field H_z plays the role of the potential function for the electric field. Furthermore, the fields have a relative phase shift of $\pi/2$.

The fact that the resonance electric and magnetic fields are successively concentrated in different bounded regions makes it possible to regard the cylinder with the slit as a high Q circuit with lumped parameters. The edges of the slit play the role of the capacity, while the walls of the cylinder play the role of the inductance.

An analysis of the equiphase lines and the lines of the mean energy flux averaged over a period near the cylinder with a slit (Fig. 3) shows that at the resonance frequency the phase front of the incident wave is highly distorted; phase nodes appear, in whose vicinity energy

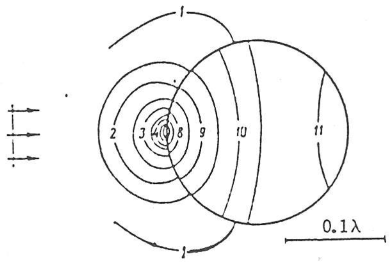

FIG. 2. Structure of the resonance field. Lines H_z = const for $\varphi_0 = 180°$, $\theta = 5°$, $ka = 0.375$.

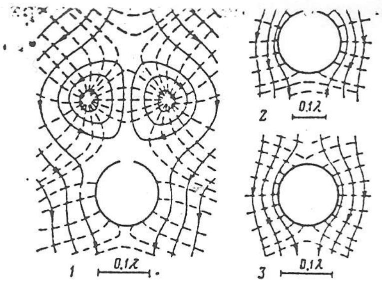

FIG. 3. Equiphase lines (dashed lines) and lines of mean energy flux near the cylinder with slit in the resonance and nonresonance case. 1) $ka = 0.375$, $\theta = 5°$; 2) $ka = 0.6$, $\theta = 5°$; 3) $ka = 0.375$, $\theta = 0$.

circulation occurs along closed trajectories. Inside the cylinder the energy flux averaged over a period is zero, since in the regime of stationary oscillations the energy is only stored successively in the form of the electric field around the slit and in the form of the magnetic field inside the cylinder.

The phenomenon investigated here is clearly distinguishable even for very wide slits (Fig. 1); however, it loses its longwave character. For example, for $\theta = 90°$, the fact that approximately one half-wavelength of the incident wave is contained in the arc length of the transverse section of the half cylinder.

In conclusion we note that the resonance regime of the cylinder with the slit, described above, is all the more interesting in view of the fact that the electromagnetic analog of a Helmholtz resonator does not exist for the classical structure in the form of a sphere with an aperture. This is a consequence of the vector nature of the electromagnetic oscillations.

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On the origin and structure of the Lorenz attractor

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One of the urgent problems which arose in Boltzmann's investigations is the resolution of the following question: Can physical processes which are quite definitely "stochastic" in nature be explained within the framework of a dynamic description without involving additional hypotheses using the ideas of probability theory? Interest in this problem has recently been especially strengthened, as can be explained on the one hand by modern advances in those areas of the qualitative theory of multidimensional systems which are based on the coarseness concept and on the other hand by the construction of new dynamic models of real processes which cannot be referred to traditional problems of the theory of vibrations since they have, in principle, new kinds of motions – homoclinic Poincaré curves. The mere existence of a homoclinic curve, involving the existence of a countable set of periodic motions, a Poisson-stable continuum, etc.,[1,2] still does not denote "stochasticity": It is necessary that homoclinic curves enter into attracting sets, the so-called "strange attractors."[5] Hence, the problem of the mathematical structure of attractors occurs naturally, since the question of the adequacy of a model for a physically observable stochastic process is related to them namely. It is known from the theory of hyperbolic sets that Y-systems, Williams solenoids, etc., can be strange attractors in coarse systems. However, the possibility of their appearance in model systems still remains problematical.

The purpose of this paper is to solve this problem in two specific cases which occur in hydrodynamics and nonlinear optics. The first model is associated with an attempt to explain the turbulence phenomenon of convective fluid motion by using the system

$$\dot{x}=-\sigma(x-y), \quad \dot{y}=-xz+rx-y, \quad \dot{z}=xy-bz, \tag{1}$$

derived by Saltzman[3] from the Navier–Stokes equations by using a Galerkin procedure. Lorenz[4] detected the nonperiodic nature of the trajectories[1)] in this model by computer computation.

The second model,

$$\dot{x}=-\gamma_3(x-y), \quad \dot{y}=\gamma_2(xz-y), \quad \dot{z}=-\gamma_1(z-z_0)+\gamma_1(z_0-1)xy, \tag{2}$$

has been constructed to explain the generation of vibrations in a laser (see the survey, Ref. 6). Hence, it is interesting to note that despite the distinct physical nature of the problems under consideration, the systems (1) and (2)

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