# Near and Far Fields of High-Quality Resonances of an Infinite Grating of Sub-Wavelength Wires

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***Abstract*—An infinite grating of identical dielectric cylinders placed periodically in free space and illuminated by a plane wave is investigated. Numerical and analytical solutions are derived for the E and H-polarisation cases. The study is focused on the ultra high-Q resonances at the frequencies close to the Rayleigh "anomalies" where the Floquet harmonics "pass over horizon." The in-resonance near field patterns for the scattering problem are demonstrated and compared to the natural mode fields. They reveal that, near the grating, the dominant contribution comes from two Floquet harmonics propagating along the plane of grating however evanescent in the normal direction. As a result, a powerful standing wave is formed along the grating, with the amplitude proportional to the natural mode Q-factor.**

***Keywords- grating, sub-wavelength wires, reflection resonances, near field, Q-factor, lasing threshold.***

## I. INTRODUCTION

Transmission and reflection of plane waves by periodic gratings (grids) of circular dielectric wires have been studied in the numerous papers since the late 1950s (for instance, see [1-3]) as one of the canonical scattering problems. Therefore it appears amazing that the most interesting resonances taking place near the Rayleigh "anomalies" associated with spatial field harmonics "passing over horizon" were first noticed only recently [4], through the approximate analytical study. The reason for overlooking them can be seen in the extremely high-Q character of these resonances, especially for sparse gratings. In this paper, we reveal their nature and demonstrate that they form doublets, where each resonance is caused by the presence of high-Q natural mode of the grating whose field possesses either symmetry or anti-symmetry across the grating plane.

## II. FORMULATION AND BASIC EQUATIONS

Consider a grating made of the parallel to the *z*-axis and periodic along the *x*-axis circular cylinders in free space – see Fig. 1. Denote the period as $p$, the wire radius as $a$, and the refractive index of the wires as $\nu$. We suppose that the electromagnetic field is time-harmonic $\sim \exp(-i\omega t)$ and does not vary along the *z*-axis. Then two alternative polarisations, $E$ and $H$, can be considered using one function $U$ that is either the $E_z$ or the $Z_0 H_z$ component of the electromagnetic field, depending on the polarisation.

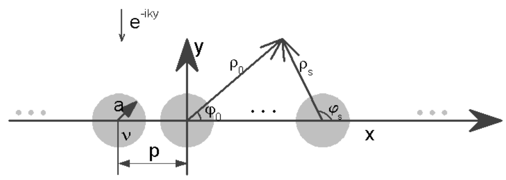


Fig. 1 Infinite periodic grating of circular cylinders illuminated by a normally incident plane wave.

This function must satisfy the Helmholtz equation with appropriate wavenumber inside and outside of cylinders, the Sveshnikov radiation condition [5] at infinity, condition of local integrability of power, and the boundary conditions demanding continuity of the tangential field components at the cylinder boundary. The free-space wavenumber is $k = \omega / c = 2\pi / \lambda$, where *c* is the light velocity and $\lambda$ is the wavelength, while inside the cylinders it is $k\nu$.

We will consider the normal incidence of a plane wave. Then, according to the Floquet theorem, the scattered field is a periodic function of *x* with period *p*. In this case we can investigate the field within one elementary cell of the grating and use the boundary conditions on one cylinder. Here, the circular shape of the wire cross-section suggests that the field can be expanded in terms of the azimuth angle in the polar coordinates co-axial with the wire,

$$U^{int}(x,y) = \sum_{n=-\infty}^{\infty} a_n J_n(k\nu r) e^{in\varphi} \quad (1)$$

$$U^{ext}(x,y) = \sum_{n=-\infty}^{\infty} b_n \left[ H_n^{(1)}(kr) e^{in\varphi} + \sum_{l=-\infty}^{\infty} S_{n-l}(kp) J_l(kr) e^{il\varphi} \right] \quad (2)$$

$$S_{2q}(kp) = 2\sum_{s=1}^{\infty} H_{2q}^{(1)}(skp), \quad S_{2q+1}(kp) = 0 \quad (3)$$

Performing mathematical operations similar to [1-3] we derive an infinite matrix equation for the field expansion coefficients $b_n$. However, unlike [1-3] we emphasize that this equation has to be cast to the so-called Fredholm second kind form. This is achieved by re-scaling the unknown coefficients as $b_n = x_n J_n(ka)$, so that we obtain

$$x_m + \sum_{n=-\infty}^{\infty} x_n \frac{S_{n-m}(2\pi\sigma)V_m(u,\nu)J_n(u)}{F_m(u,\nu)J_m(u)} = \frac{(-1)^m V_m(u,\nu)}{F_m(u,\nu)J_m(u)}, \quad (4)$$

$$F_m(u,\nu) = \nu^{\pm 1} H_m(u) J'_m(\nu u) - H'_m(u) J_m(\nu u), \quad (5)$$

$$V_m(u,\nu) = \nu^{\pm 1} J_m(u) J'_m(\nu u) - J'_m(u) J_m(\nu u), \quad (6)$$

where we denote $u = 2\pi\sigma/\xi$, $\sigma = ka\xi/2\pi = p/\lambda$, $\xi = p/a$, and the signs "±" correspond to the $E$ and $H$ - case, respectively.

This procedure is crucially important because only the Fredholm form guarantees the convergence of solutions of the truncated equation to exact one if the order of truncation gets greater.

Note that $S_l(2\pi\sigma)$ are the lattice sums depending only on the electrical period of the grating. These series converge very slowly and therefore acceleration procedures are necessary - see details in [6]. The functions $F_m$ and $V_m$ depend on the parameters of one cylinder only. Outside of the grating domain, which is the strip of the width 2*a*, the scattered field can be conveniently converted to the series in terms of the Floquet spatial harmonics by applying the Poisson summation formula and the Sommerfeld integral representation for the Hankel functions,

$$U^{ext,\pm}(x,y) = \sum_{s=-\infty}^{\infty} f_s^{\pm} e^{ik\pi_s x} e^{\pm ik\gamma_s y}, \quad (7)$$

$$f_s^{\pm} = (\pi\sigma\tau_s)^{-1} \sum_{n=-\infty}^{\infty} x_n J_n(ka)(-i\pi_s \pm \tau_s)^n, \quad (8)$$

where $\pi_s = s/\sigma$, $\tau_s = \sqrt{1-\pi_s^2}$. If $p < \lambda$, then the reflectivity of the grating is $R = f_0^+$, and the power conservation law implies that $|f_0^+|^2 + |f_0^- + 1|^2 = 1$.

Besides of the plane wave scattering, it is also interesting to study the corresponding eigenmode problems. We consider them, for the both polarisations, in a modified formulation that is adapted to the analysis of lasing [7,8]. This means that we assume that the wire material is "pumped" (i.e. displays negative absorption) and characterise it with the aid of the negative imaginary part of refractive index, i.e. set $\nu = \alpha - i\gamma$. A grating made of such wires (so-called quantum wires) is able to balance the radiation losses of an eigenmode with the power generated in the wires provided that the material gain $\gamma > 0$ reaches the threshold value. In the case of eigenproblem, we assign the right–hand part of (4) to be zero and solve the determinantal equation, $\det[I + A(\sigma;\xi,\nu)] = 0$, where the matrix $A$ is the same as in (4) however $F_m$ and $V_m$ are now the functions of a complex argument. The roots are the pairs of real numbers $(\sigma,\gamma)$, namely the normalised frequencies of lasing modes and their thresholds. As it was noted in [9], if $\xi \to \infty$ the frequencies of the resonances associated with the periodicity tend to $\sigma = 1, 2, \ldots$ and their Q-factors grow up. If the wires are pumped, the associated lasing thresholds of these modes get down for $\xi \to \infty$.

## III. Near Field Behavior

Consider a relatively dense grating of wires with $\nu = 1.4$ and the period-to-radius ratio of $\xi = 2.2$. Fig. 2 (a) shows the frequency scan of the reflectivity of the grating for the normal incidence of the E-polarised plane wave, with two clearly observable resonances. The first of them, at the frequency $\sigma = 0.8256$, corresponds to the symmetric eigenmode $E_1^+$, as seen from the pattern of the near field in the scattering problem Fig. 2 (b) and the eigenmode field shown in Fig. 2 (c). The second resonance, $E_1^-$, which sits at $\sigma = 0.98365$, has anti-symmetric field with respect to the grating plane, see Figs. 2 (d) and (e). The fields are visualised using the code from [10].

The H-polarisation case (Fig. 3) shows similar resonances $H_1^{\pm}$,

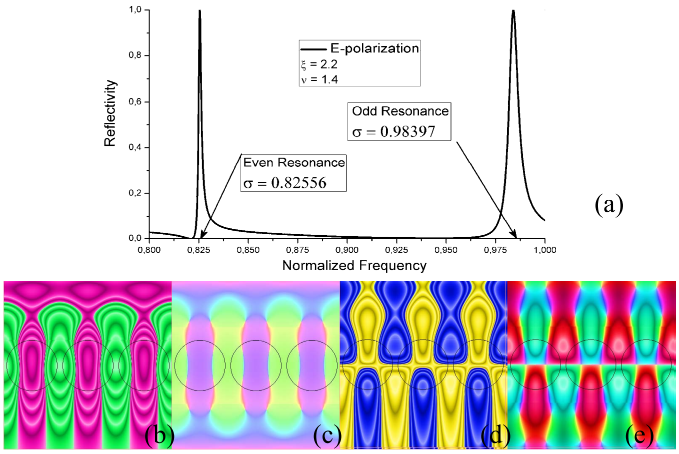


Fig. 2 Frequency scan of the E-polarisation reflectivity for the grating with $\xi = 2.2$ made of circular dielectric wires with $\nu = 1.4$ (a). The near field in the even-mode resonance at the frequency $\sigma = 0.8256$ (b) and corresponding eigenfield for eigenvalues $\sigma = 0.8253$ and $\gamma = 0.00175$ (c). The near field in the odd-mode resonance at $\sigma = 0.98365$ (d) and corresponding eigenfield for $\sigma = 0.98367$ and $\gamma = 0.0124$ (e).

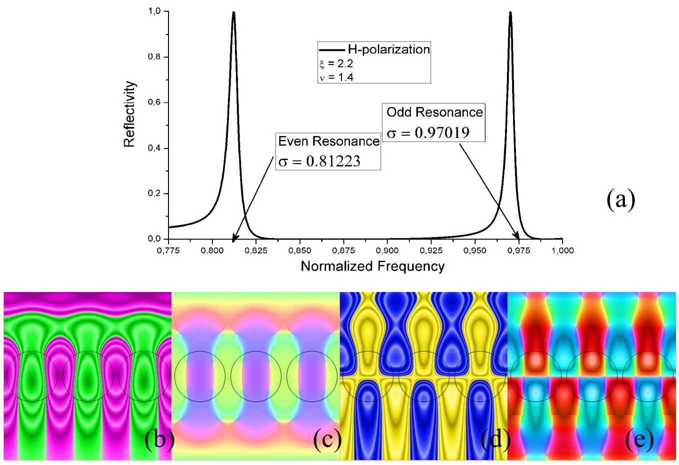


Fig. 3 The same as in Fig. 2 however for the H-polarisation. The even-mode resonance field at $\sigma = 0.81223$ (b) and its eigenfield $\sigma = 0.8140$ and $\gamma = 0.007$ (c). The second odd resonance field at $\sigma = 0.97019$ (d) and the corresponding eigenfield for $\sigma = 0.9701$ and $\gamma = 0.00877$ (e).

whose Q-factors are higher. Because of this, further we consider only the H-polarization.

In the case of the quantum-wire grating able to emit the radiation under pumping, the grating modes display rapid decrement of the lasing thresholds if the grating gets sparser.

This is demonstrated in Fig. 4, where the convergence of the lasing-mode $H_1^+$ frequency to $\sigma = 1$ is also shown.

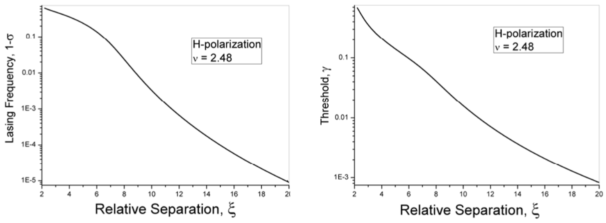


Fig. 4 The lasing eigenvalues for the grating with refractive index $\nu = 2.48$ and varying $\xi$: the normalized lasing frequency (left) and the corresponding threshold gain (right).

The most amazing feature of the grating-type resonances $H_m^\pm$ in the plane-wave scattering is the near field behaviour. For instance, if $\xi = 10$ we observe a narrow full-reflection resonance, see Fig. 5, and the near field displays two kinds of well-shaped standing waves.

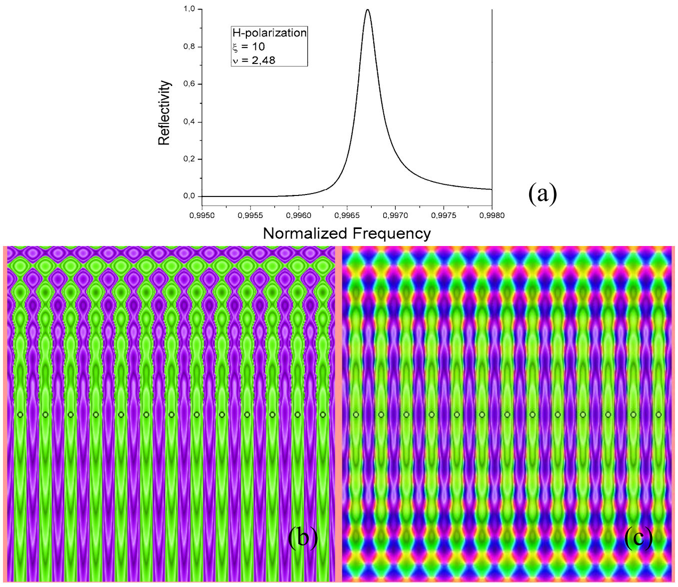


Fig. 5 H-polarization: (a) the frequency scan of the reflectivity for the grating with $\xi = 10$, $\nu = 2.48$, (b) the near field in resonance at $\sigma = 0.99671217$, and (c) the corresponding eigenfield with $\sigma = 0.99669$ and $\gamma = 0.01595$ (19 periods both).

The first of them appears normally to the grating in the reflection halfspace (domain I). It is built from the incident wave and the 0-th Floquet harmonic $f_0^+ e^{-iky}$ whose amplitude is close to 1. The second standing wave appears in domain II stretched along the grating plane and has maxima of intensity inside the cylinders and right in the middle between them. It is built from the ±1-st Floquet harmonics and therefore has the form $U(x,y) \approx 2 f_1^+ e^{ik\tau_1 y} \cos(k\pi_1 x)$. Note that the amplitude $f_1^+$ is not restricted by the power conservation law because corresponding harmonics decay in the normal direction ( $\tau_1^2 < 0$ ). Domain III, in the transmission half-space, is dark. The width of domain II, where the total field is close to the eigenmode field, is $|y| \le 5p$.

Fig. 6 shows the same data for a twice sparser grating, for $\xi = 20$. The resonance in the reflectance can be found here only if, in computations, the step in $\sigma$ is $10^{-6}$ or smaller. One can see the near-field pattern in Fig. 6 (b) similar to Fig. 5 (b) however even the visualisation window of 33 periods does not show domain I. Only the standing wave along the grating plane is clearly seen, and the near field for the scattering problem is the same as for the eigenmode.

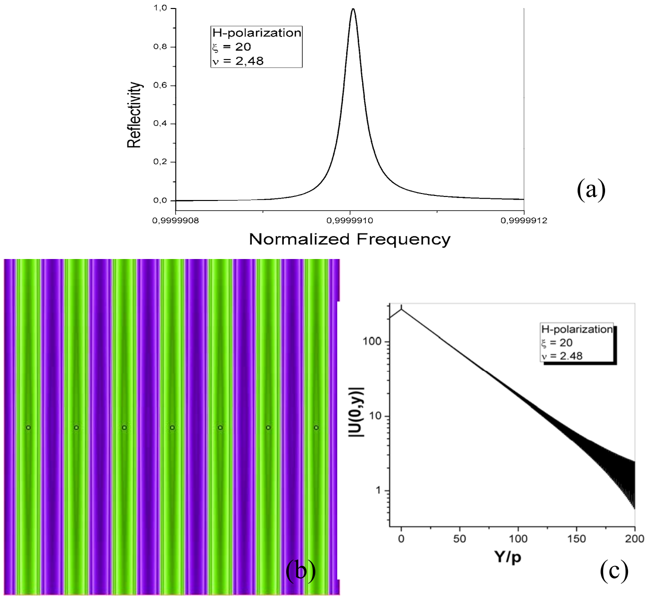


Fig. 6 H-polarization: (a) the frequency scan of the reflectivity for the grating with $\xi = 20$, $\nu = 2.48$, (b) the near field in resonance at $\sigma = 0.999991$ (7 periods), and (c) the cut of the near field along the $y$-axis.

The lasing frequencies and threshold gains of the grating modes $H_m^\pm$, can be obtained analytically if $\xi \to \infty$,

$$(9)\ \sigma_m^{H+} = m - \frac{2\pi^4 m^3 (1-\nu)^2}{\xi^4}, \gamma_m^{H+} = \frac{\pi^3 m(\pi m - 1)(1-\nu)^2}{4\xi^2},$$

$$(10)\ \sigma_m^{H-} = m - \frac{\pi^4 m}{2\xi^4}\left(\frac{\nu^2-1}{\nu^2+1}\right)^2, \gamma_m^{H-} = \frac{\pi(\pi m - 2)\sqrt{m}}{2\sqrt{2}\nu\xi^2}\frac{\nu^2-1}{\nu^2+1}.$$

This is in agreement with numerical data of Fig. 4. A similar treatment of the scattering-problem eigenvalue in terms of the complex frequencies and associated Q-factors results in the following estimations:

$$Q_m^{H+} = \frac{m\xi^2}{2\pi^2(m\pi - 1)(\nu - 1)}, \quad Q_m^{H-} = \frac{m\xi^2}{\pi^3(m\pi - 2)} \frac{\nu^2 + 1}{\nu^2 - 1} \quad (11)$$

These formulas tell that the Q-factors grow and thresholds decay if the contrast in refractive indices between the wire material and the host medium gets smaller. Indeed this is demonstrated by the near-field patterns in Fig. 7 where again the total field in domain II is identical to the eigenfield of the natural lasing mode.

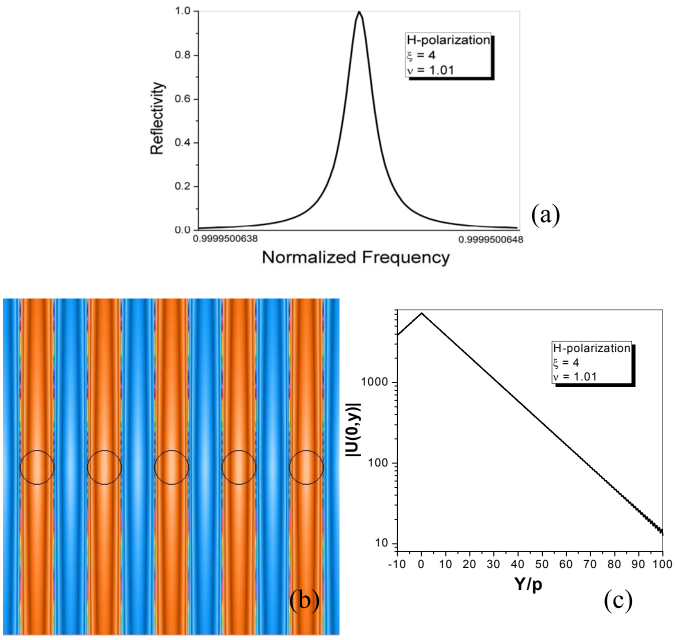


Fig. 7 H-polarization: (a) the frequency scan of the reflectivity for the grating with $\xi = 4$, $\nu = 1.01$, (b) the near field in resonance on $\sigma = 0.99995035252$ (5 periods), and (c) the cut of the near field along the $y$-axis.

The analysis of equations (1)-(8) shows that in the reflection resonance associated with the grating mode $H_1^+$ the amplitude $f_1^+ = const \cdot Q_1^{H+}$. Therefore the scattered field has the form, in resonance, as

$$U(x, y; \sigma_1^{H+}) \approx const \cdot (1/\delta) e^{-\delta k|y|} \cos[k(1+\delta^2)x], \quad (12)$$

where $\delta = 2\pi(\nu - 1)/\xi^2 << 1$. This leads to conclusion that the width of domain II of the dominance of the standing wave (12) over the incident-wave field can be determined from the condition

$$|y|/p \approx (2\pi\delta)^{-1} \ln(1/\delta) \quad (13)$$

This yields $|y| \approx 225p$ in the case depicted in Fig. 7 (b) where $\delta = 3.9 \cdot 10^{-3}$ that agrees well with the actual field behavior presented in Fig. 7 (c). The presence of losses in the wire material should, however, make these remarkably large values smaller.

## IV. Conclusion

We have studied the ultra high-Q resonances in the plane-wave scattering by a dielectric-wire grating at the frequencies close to the Rayleigh "anomalies" where Floquet harmonics "pass over horizon." Besides, we have found the associated eigenvalues in the H-polarization case analytically and numerically. The in-resonance near field patterns for the scattering problem have been studied and compared to the natural-mode fields. Their nature has been explained via the standing waves built from (i) the incident and the specularly reflected plane waves along the $y$-axis in the reflection half-space, and (ii) a pair of Floquet harmonics propagating in the opposite directions along the $x$-axis but exponentially decaying along the $y$-axis. The amplitude of the latter is proportional to the Q-factor of the in-resonance grating mode and the domain of its dominance stretches to $O(-\sqrt{Q}\ln Q)$ periods above and below the grating.


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