Asymptotic Approximations of Lasing Eigenvalues of an Infinite Grating of Circular Quantum Wires in the Free Space

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***Abstract*— One of fundamental problems of electromagnetics is the scattering of a plane wave from a grating of infinite circular dielectric cylinders (wires). Lord Rayleigh was apparently one of the first researchers who formulated it in the XIX century. We consider the associated lasing eigenvalue problem and focus our attention on the asymptotic approximations of the eigenvalues for large values of the ratio of the grating period to the wire radius. Classification of the modes located near the Rayleigh anomalies is given.**

Keywords—grating; eigenvalues; lasing;

I. Introduction

Although the scattering of a plane wave by a periodic grating of circular cylinders or wires is a fundamental problem of electromagnetics and has been profoundly studied since the 1960s [1], it was only in [2] where specific resonances appearing near Rayleigh anomalies were studied in the reflection and transmission coefficients. In the earlier works [3-5], it was proved that these resonances are associated with so-called "grating modes." This was linked to the fact that they directly originate from singularities of the lattice sums, see [6-8]. They exist also on the finite-size gratings of wires and strips - see [9,10] and [11,12], respectively. A review on these modes can be found in [13]. For sub-wavelength wires, the poles corresponding to these modes cluster near the Rayleigh anomalies. In this work, we derive the asymptotic expressions for such modes.

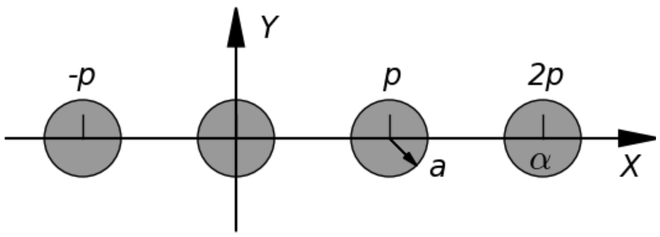

Fig. 1. Sketch of geometry of a grating of dielectric circular cylinders in the free space.

II. Problem formulation

We investigated an infinite grating shown Fig. 1, having the period p, of identical parallel circular dielectric cylinders (wires) with radius a and refractive index α. The relative separation between the wires is denoted as $\xi = p/a$ Note that this structure is symmetric with respect to both *x*-axis and *y*-axis. To study the natural states of electromagnetic field (i.e. the modes) in the presence of such grating, we considered two different formulations.

A. Active Grating Problem

Assume that the wires are so-called *quantum wires* made of the gain material able to produce the electromagnetic waves, under pumping. This enables on to compensate for the radiation losses. The gain is introduced through a complex-valued refractive index of wire material, where the imaginary part is negative. Then we can consider eigenvalues as pairs of two real numbers: the real-valued normalized frequency $\sigma = ka\xi / 2\pi = p/\lambda$ (λ being the wavelength) and the lasing threshold or imaginary part γ of the wire refractive index $\nu = \alpha - i\gamma$. Details of this approach can be found in [3,14].

B. Passive Grating Problem

The eigenvalue formulation is more classical: we consider the wires with real-valued refractive index $\nu = \alpha$ and assume a complex-valued frequency, $\sigma = \sigma' - i\sigma''$. Here the real and imaginary parts of frequency are considered as eigenvalues.

III. Derivation of Analytical Formulas

According to the Floquet theorem, the scattered field is a periodic function of x with period p. In this case we can investigate the field within one elementary cell of the grating and use the boundary conditions on one wire. Here, the circular shape of the wire cross-section helps expanding the field in terms of the azimuth angle in the polar coordinates co-axial with the wire. Performing standard mathematical operations, we derive an infinite matrix equation for the field expansion coefficients. However, we emphasize that,

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unlike[1,2,6-8], this equation has to be cast to the so-called Fredholm second kind form like in [3-5].

$$\det\left(A\right)=0,\ A_{n,m}=\delta_n^m+\sum_{n=-\infty}^{\infty}\frac{S_{n-m}(2\pi\sigma)V_m(u,\nu)J_n(u)}{F_m(u,\nu)J_m(u)}, \quad (1)$$

where

$$F_m^{E,H}(u,\nu)=\nu^{\pm 1}H_m(u)J'_m(\nu u)-H'_m(u)J_m(\nu u), \quad (2)$$

$$V_m^{E,H}(u,\nu)=\nu^{\pm 1}J_m(u)J'_m(\nu u)-J'_m(u)J_m(\nu u), \quad (3)$$

are the functions of wire parameters, ν^{+1} corresponds to the E-polarization and ν^{-1} to the H-polarization, and S_n are the lattice sums (introduced in [8])

$$S_{2q}(\sigma)=2\sum_{s=1}^{\infty}H_{2q}^{(1)}(s\sigma),\quad S_{2q+1}(\sigma)=0 \quad (4)$$

$$S_0=-1-\frac{2i}{\pi}(\gamma+\log\frac{\sigma}{2})-\frac{2i}{\sigma\tau_0}+\sum_{s\neq 0}\left(\frac{i}{\pi|s|}-\frac{2i}{ka\tau_s}\right) \quad (5)$$

$$S_n=\sum_{s=-\infty}^{\infty}\frac{[\pi_s-\mathrm{sgn}(s)\tau_s]^n}{i\tau_s}+\frac{2i^{n+1}}{n\pi}\cos\left(n\pi/2\right)$$
$$+\frac{i^n}{\pi}\sum_{j=0}^{[(n-1)/2]}(-1)^j\frac{(n+j-1)!}{j!(n-2-j)!}\left(\frac{4\pi}{ka}\right)^{n-2j}B_{n-2j} \quad (6)$$

where B_n are Bernoulli coefficients, $\pi_s=s/\sigma,\quad \tau_s=\sqrt{1-\pi_s^2}$. Note that the lattice sums have singularities if the argument equals to an integer number.

According to the Gerschgorin theorem [15], if a determinant equation satisfies the second kind Fredholm condition, there exist vicinities of the zeros of the diagonal elements, or roots of $A_{m,m}(\sigma,\nu,\xi)=0$, where exact roots of the whole determinantal equation lie. Therefore consider such roots of the m-th diagonal element of (1), related to the y-even grating mode,

$$1+\frac{S_0(2\pi\sigma)V_m(u,\nu)}{F_m(u,\nu)}=0 \quad (7)$$

For large values of ξ, $V_l/F_l\xrightarrow[\xi\to\infty]{}0$ for any order l while the lattice sums $S_{2m}(2\pi\sigma)\xrightarrow[\sigma\to n]{}\infty$ for any integer value of $n>0$ which is de facto the Rayleigh anomaly. This behavior yields a finite limit for the product of two values. Therefore for any Rayleigh anomaly ($n>0$) we have an infinite number of roots of orders $m\geq 0$.

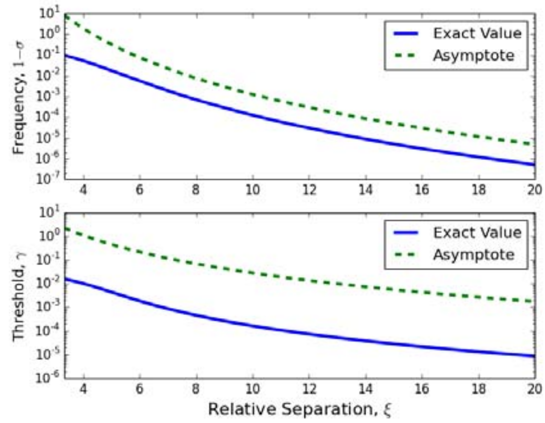
IV. Numerical Results and Asymptotics

Asymptotic expressions for the eigenvalues can be found from (1) after expanding the elements of the series in terms of powers of ξ. The results of this analysis are shown in Table I for both the active and the passive grating configurations.

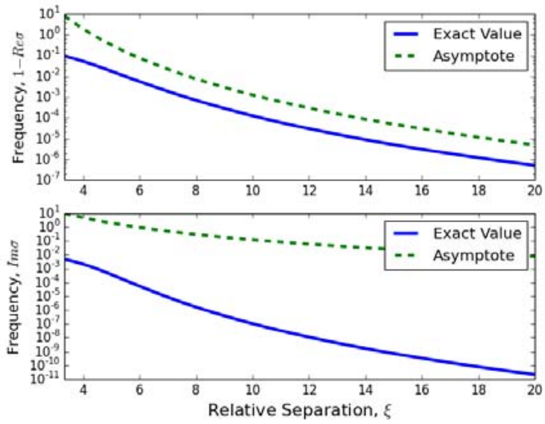
For obtaining numerical solution of (1) as a function of varying ξ, we used the Newton method with initial values at each step taken from the previous one. The starting values were taken from the scattering problem presented in [4,5].

TABLE I
Leading Terms of Asymptotics of Eigenvalues of the 0-th order (m=0) Located near the First Rayleigh Anomaly (n=1)

	H-polarization	E-polarization
σ	$1-\frac{\pi^8\left(\alpha^2-1\right)^2}{2\xi^8}$	$1-\frac{\pi^4 1\left(\alpha^2-1\right)^2}{2\xi^4}$
γ	$\frac{\pi^4\left(1-\alpha^2\right)^2\left(\pi-1\right)}{8\alpha\xi^4}$	$\frac{\pi^2\left(1-\alpha^2\right)^2\left(\pi-1\right)}{8\alpha\xi^2}$
σ'	$1-\frac{(1-\alpha^2)^2}{2}\frac{\pi^8}{\xi^8}$	$1-\frac{(1-\alpha^2)^2}{2}\frac{\pi^4}{\xi^4}$
σ''	$-\frac{(\alpha^2-1)\pi^4}{(\pi-1)\xi^4}$	$-\frac{(\alpha^2-1)\pi^2}{(\pi-1)\xi^2}$

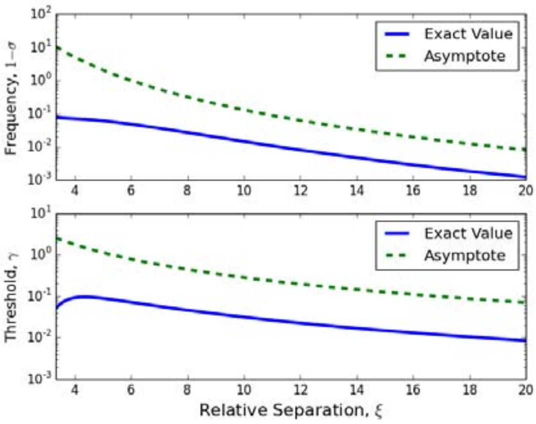


(a)

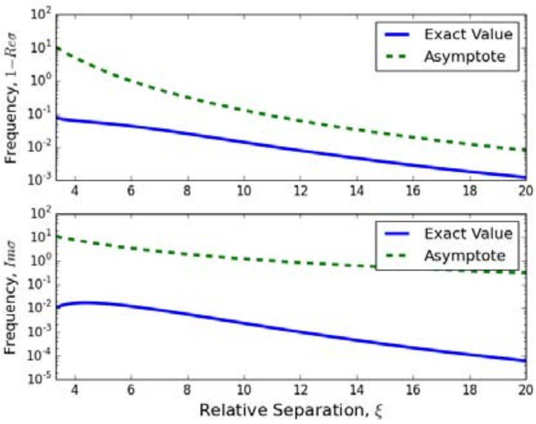


(b)

Fig. 2 Exact and asymptotic eigenvalues of the y-even grating mode in the active grating problem (above) and passive one (below) for a grating in the free space made of circular wires of refractive index 2.46 depending on the separation parameter varying from 3.33 to 20 in the H-polarization.



(a)



(b)

Fig. 3 The same as in Fig. 2 however for the *E*-polarization.

In the case of the active grating problem, real valued frequency was initially taken as the resonance frequency of the reflection coefficient for corresponding geometry in the considered frequency range, and the value of lasing threshold was chosen randomly. In the case of the passive grating problem, similarly, the initial value of the real part of the eigenfrequency was taken as the resonance frequency while the imaginary part was taken randomly

The obtained exact eigenvalues are presented and compared to corresponding asymptotics in Fig. 2 for the *H*-polarization and in Fig. 3 for the *E*-polarization. Here the plots are built in semi-logarithmic scale to emphasize the rate of the frequency approach to Rayleigh anomaly and that of the threshold decay.

Both the formulas from Table 1 and the numerical plots show two interesting points.

Firstly, the eigenfrequency in the active grating problem and the real part of eigenfrequency in the passive grating problem at $\xi \to \infty$ tend to the value $\sigma = 1$, which is the Rayleigh anomaly. Secondly, in both polarizations, the lasing threshold γ in the active grating problem, as well as the imaginary part of the eigenfrequency in the passive grating problem tend to zero at larger values of the relative separation ξ. This paradoxical behavior (the lasing properties of the grating improve if it gets sparser) should be understood properly, as an effect related to the infinite grating. In the finite gratings, such a decrement of the threshold should be limited by a value depending on the number of quantum wires.

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