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S 2

Chapter 9

Green's Function-Dual Series Approach in Wave Scattering by Combined Resonant Scatterers

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1. INTRODUCTION

1.1 Historical Background

As is well known in general, rigorous scattering and diffraction theory is based on the theory of boundary-value problems, therefore the progress here is closely tied to that of this branch of mathematics. So, as two-dimensional wave scattering problems can be reduced to singular integral equations by using a generalized potential theory approach, the Fredholm's theory of integral equations is of great importance [1]. Besides, the theory of Fourier transforms and functions of complex variable enables one to solve integral equations of a certain class [2]. This approach is called the Wiener-Hopf (WH) technique; one delivers exact solutions of such canonical diffraction problems as that for a semiinfinite zero-thickness plate, and produces effective approximate solutions for a great number of modified geometries. All these problems are known to be rearrangable in the form of equivalent dual integral equations.

The development of the Riemann-Hilbert Problem (RHP) approach can be considered as another great event in diffraction theory, although it is still less known than WH one. This technique delivers exact solution of certain dual series equations. In the core of the approach there lies a problem about reconstruction of an analytical function $X(z)$ of complex variable $z = x + iy$ whose limiting values $X^{\pm}(z)$ from inside and outside of a closed finite or infinite curve L (in) satisfy the following condition

$$X^{+}(z_0) - A(z_0)X^{-}(z_0) = B(z_0), \quad z_0 \in L \tag{1}$$

with known functions $A(z_0)$ and $B(z_0)$ called the coefficient and the free term of RHP, respectively.

Actually, the solution of this problem was first given by Carleman [3] but thorough investigation of the RHP theory and general solution of (1) for L being an arbitrary curve can be found in the books by Muskhelishvili [4] and Gakhov [5]. Agranovich, Marchenko and Shestopalov [6] were the first who applied this approach to particular scattering problem

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of plane waves diffraction by a plane-strip periodic grating. They discretized it in terms of Floquet-Fourier series expansion, reduced to the dual series equations for unknown coefficients, applied RHP solution of [3-5], and obtained an infinite Fredholm system of linear algebraic equations of the second kind. The principal point of derivations was in taking analytically all the integrals for matrix elements due to the fact that here contour L is simply a circle. As a result all the scattering characteristics can be obtained with desired accuracy through a fully correct and simple numerical procedure. Thus unlike the WH the RHP approach is of combined analytical-numerical nature implying both the analytical inversion of a certain part of the initial operator and the highly-efficient numerical inversion of the remaining part.

Thereafter this approach has been expanded upon a wide class of periodic zero-thickness $2-D$ scatterers with perfect conductivity (see Refs.[7–25]). Among them there are multielement and multilayer strip gratings (Tretyakov, Litvinenko, Kazansky, Podolsky et al., 1963–75), strip irises in a waveguide (Shcherbak, 1965–75), periodically slitted circle (Tretyakov, Sologub et al., 1967), circumferentially and helically slitted circular waveguides (Agranovich, Sologub, 1964–67), open circular screen (Koshparionok, 1972, Nosich, 1978, Ziolkowski, 1984), infinite grating of circular screens (Veliev, 1977), finite number of circular screens (Melezhik, Veliev, Veremey, 1980–86), slitted infinite cone (Sologub, Doroshenko, 1985–88), open resonators (Poyedinchuk, Brovenko, 1980–90), open strip/slot lines on circular substrates (Nosich, Svezhentsev, 1980–89). Due to the periodicity of boundaries all these problems can be rearranged in the form of dual series equations with the set of functions $\exp(in\varphi), n = 0, \pm 1, \ldots$ as a kernel (so-called trigonometric kernel due to the known representation $\exp(in\varphi) = \cos n\varphi + i \sin n\varphi$). Consequently, there appears an RHP with the contour L being a unit circle L^0.

As for the most impressive amount of fundamental and applied results, in '60s and '70s they were obtained in the analysis of diffraction by strip gratings. However by '80s the focus has been shifted to a circular nonclosed screen and collections of such screens. Recently, it has been shown that numerical analysis of the $2-D$ scattering from an arbitrary smooth open screen can be efficiently reduced to canonical (circular) case by appropriate parametrization [26].

Selected works in this field until 1986 were duplicated in the books by Shestopalov [27–30] (in Russian). Relevant problems were analysed by Litvinenko and Salnikova [31]. Ziolkowski [32, 33] was the only author in the West who exploited this approach. All the solutions mentioned above can be considered as "numerically rigorous" because they are not fully analytical but provide the results with any degree of approximation. The restrictions are of purely computational nature as one has to store and process a matrix equation of the order somewhat greater than characteristic dimension to wavelength ratio.

1.2 On the Extention of the Range of Applications.

One can conclude that probably all canonical structures treatable by conventional RHP technique have already been analyzed. Further development leading to the extention of the range of applications requires some new ideas. One of those can be seen in combining conventional RHP method with other classical methods for the analysis of combined structures. For example, although a free-space diffraction is of great value both for microwave theory and applications, it is often desirable to take into account correctly the real inhomogeneous environment of the scatterer. An important class of such problems connected with a plane-parallel layered dielectric medium with imbedded localized obstacles arises. As for the open screen, examples can be found in modelling a reflector antenna radiation in the presence of an interface between two media, dielectric-slab modes scattering and conversion by screen-like inhomogeneities, excitation of and energy leakage from a multiple-mirror open resonator containing a material slab, etc. Another related class of scattering problems arises for screens near infinite plane-periodic boundaries, i.e. near periodic diffraction gratings. Such problems are generated by the necessity of simulation of open resonators with grating inhomogeneities for millimeter-wave oscillators and spectroscopic cells.

So far, all these problems have never been analysed correctly in full-wave dynamic formulation. The aim of this treatment is to demonstrate that they are solvable by the RHP approach as well, provided that this method is combined with the Green's function technique. It means that one must firstly construct the Green's function of $2-D$ space containing all the elements except the screen, and further apply the generalized potential theory and RHP-based inversion procedure.

As for a dielectric-layered medium, the $2-D$ Green's functions are available in explicit form as Fourier-type integrals with exactly known transforms. For a space with a grating the Green's functions are not available analytically, but nevertheless for many particular gratings there exist effective numerical approaches in Fourier transform domain. Those are known for gratings of plane or inclined strips, of semi-infinite planes, of circular or rectangular rods, for echelette etc., and solutions can be found with any desired accuracy.

Due to infinite boundaries of a slab or a grating the formulation of mentioned problems has to be changed in comparison to the free-space diffraction. The well known Sommerfeld condition of radiation is not necessarily valid here, and to ensure the uniqueness of the solution it has to be replaced by some other one adequate to the combined geometry. This question has been analysed by Nosich [34] who showed that the modified condition of radiation contains all the natural guided (undecaying) modes of a slab. The same is valid for gratings provided that such modes do exist.

2. MATHEMATICAL FOUNDATIONS OF THE METHOD

The RHP technique is a powerful tool of solving wave scattering and diffraction problems and requires knowlege of the theory of analytical functions of complex variable and Cauchy-type integrals. In this section we shall summarize main information necessary for futher sections and give a method of solution for canonical dual series equations.

2.1 About the RHP in a Complex Plane

Consider a simple smooth nonintersecting closed curve L in the plane of complex variable $z = x + iy$, and define two open domains $Q^{\pm}$ such that $Q^+ = intL$ and $Q^- = extL$ as shown in Fig.1.

Let $X(z)$ be a sectionally analytical function such that $X(z) = X^{\pm}(z)$ for $z \in Q^{\pm}$. If we assume that 1) limiting values of $X(z)$ from inside and from outside of L, namely $X^{\pm}(z_0)$, $z_0 \in L$, satisfy a transmission condition given by

$$X^+(z_0) - X^-(z_0) = B(z_0) \tag{2}$$

with at least Hölder-continuous known function $B(z_0)$ as a free term, and 2) $X(z)$ is vanishing at $|z| \to \infty$, then it is known that the whole function $X(z)$ can be expressed in the form of Cauchy integral

$$X(z) = \frac{1}{2\pi i} \int_L \frac{B(z_0)dz_0}{z_0 - z} \tag{3}$$

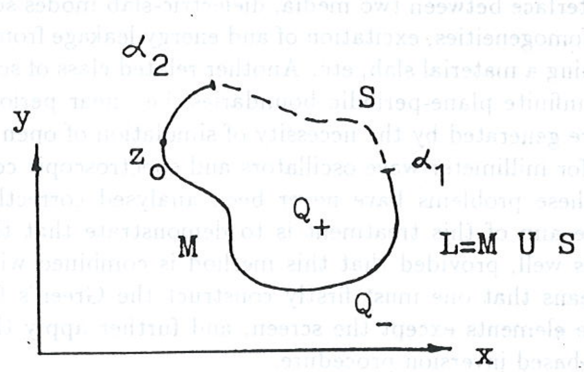

Fig. 1. Complex z-plane separated to bounded and unbounded domains by a closed curve L.

For such integrals, the Plemelj-Sokhotskii formulas are valid, giving the limiting values $X^{\pm}(z_0)$ as

$$X^{\pm}(z_0) = X(z_0) \pm \frac{1}{2}B(z_0) \tag{4}$$

The RHP is a somewhat generalized version of the problem (2), and involves into the treatment another known function $A(z_0)$ on the curve L, of the same class as $B(z_0)$, so that

$$X^{+}(z_0) - A(z_0)X^{-}(z_0) = B(z_0) \tag{5}$$

Further generalizations can be introduced by setting coefficient $A(z_0)$ to be a discontinuous function on L, and by setting L to be an open curve. Besides, the behaviour of $X(z)$ at the infinity can be not necessarily vanishing but described by a polynomial function of z. Consider a boundary value problem about the reconstruction of an analytical function $X(z)$ after the boundary expressions

$$X^{+}(z_0) + X^{-}(z_0) = B(z_0), \qquad z_0 \in M \subset L \tag{6a}$$

$$X^{+}(z_0) - X^{-}(z_0) = 0, \qquad z_0 \in S = L \backslash M \tag{6b}$$

given at the open smooth enough curves M and S completing each other to a closed curve L.

The equation (6b) states that functions $X^{\pm}(z)$ continue analytically across S. Thus, they represent the same analytical function of z in $Q^{\pm}$ respectively. The two of equations (6) can be rearranged as a single one by introducing discontinuous coefficient and free term functions

$$\tilde{A}(z_0) = \begin{cases} 1, & z_0 \in M \\ -1, & z_0 \in S \end{cases} \tag{7}$$

and

$$\tilde{B}(z_0) = \begin{cases} B(z_0), & z_0 \in M \\ 0, & z_0 \in S \end{cases} \tag{8}$$

so that one obtains

$$X^{+}(z_0) + \tilde{A}(z_0)X^{-}(z_0) = \tilde{B}(z_0) \tag{9}$$

Note that equation (9) is valid on the whole closed curve L.

To make further derivations, it is necessary to specify the behaviour of unknown function $X(z)$ at infinity and at the endpoints of the open curve M, where the functions $\tilde{A}(z)$ and $\tilde{B}(z)$ loose continuity.

We shall assume that $X(z)$ is vanishing as $|z| \to \infty$, however the functions of different behaviour, polynomial in general, can be treated by the RHP technique as well. Besides, we shall assume that $X(z)$ has singularities of the order 1/2 at each of two endpoints of M. This behaviour is typical for electromagnetic problems of wave scattering and diffraction by perfectly conducting zero-thickness screens. However, in other applications this is not true, like, e.g., in fluid and gas dynamics where the velocity potential function has no singularity at the rear edge of a thin hard wing. Basically, the RHP approach enables one to treat any singularities of the order less than 1, and not necessarily equal to each other.

2.2 Solution of an RHP Typical for Electromagnetics

Guided by the above assumptions, we can introduce a characteristic function $R(z)$ of the problem under consideration, such that the function $X(z)\ R(z)$ is regular in the whole complex plane including endpoints of M. The procedure of seeking this function is described in [4,5] and brings

$$R(z) = (z - \alpha_1)^{1/2}(z - \alpha_2)^{1/2} \tag{10}$$

with $z = \alpha_{1,2}$ for endpoints, and the branch chosen so that limits differ by sign : if $z \in Q^{\pm}$, $z_0 \in M$ then for $z \to z_o$ $R(z) \to R^{\pm}(z_0) = \pm R^{+}(z_0)$. Thus, the characteristic function satisfies a homogeneous RHP on the closed contour L

$$\frac{1}{R^{+}(z_0)} + \frac{1}{R^{-}(z_0)} = 0 \tag{11}$$

Now, by introducting new functions

$$Y(z) = X(z)R(z) \tag{12}$$

$$D(z_0) = \tilde{B}(z_0)R^{+}(z_0) \tag{13}$$

we come to an RHP with continuous coefficient function, valid on a closed curve L

$$Y^{+}(z_0) - Y^{-}(z_0) = D(z_0) \tag{14}$$

As the characteristic function has a simple pole at infinity due to the fact that $R(z) = O(z)$ as $z \to \infty$, it is known [4,5] that the solution (3) has to be modified by adding a constant C, so that

$$Y(z) = \frac{1}{2\pi i}\int_{L} \frac{D(z_0)dz_0}{z_0 - z} + C \tag{15}$$

Rewriting (15) for the function $X(z)$ we find

$$X(z) = \frac{1}{2\pi i}\frac{1}{R(z)}\int_{M} \frac{R^{+}(z_0)B(z_0)dz_0}{z_0 - z} + \frac{C}{R(z)} \tag{16}$$

Expression (16) delivers the exact solution of initial transition problem of Riemann-Hilbert (6) with mentioned assumptions about the behaviour at infinity and at the endpoints of curve M. However, this solution is not much effective computationally because of the singular character of the integral term in the right hand part. As for the constant C, its value can usually be obtained due to certain additional conditions following from the physical nature of initial problem which generates an RHP like (6).

2.3 Exact Solution of Canonical Dual Series Equations

Consider the following dual series equations with "trigonometric" kernel for infinite sequence of coefficients μ_n, $n = 0, \pm 1, \ldots$

$$\sum_{(n)} \mu_n |n| e^{in\varphi} = F(e^{i\varphi}), \qquad \varphi \in M = (\theta < |\varphi| \leq \pi) \tag{17a}$$

$$\sum_{(n)} \mu_n e^{in\varphi} = 0, \qquad \varphi \in S = (|\varphi| < \theta) \tag{17b}$$

where the sums are taken from $-\infty$ to ∞.

Assuming that the vector $\mu = \{\mu_n\}_{n=-\infty}^{\infty}$ is of the space l_2, and series in (17) are term-by-term differentiable, we replace (17b) by its derivative with respect to φ and add the identity at $\varphi = 0$, the center point the interval S. Denoting $\beta = \pi - \theta$, $\psi = \pi + \varphi$ and $F'(e^{i\psi}) = F(e^{i\psi - i\pi})$, we have

$$\sum_{(n)} \mu_n |n| e^{in(\psi-\pi)} = F'(e^{i\psi}), \qquad \psi \in M' = (|\psi| < \beta) \tag{18a}$$

$$\sum_{(n)} \mu_n n e^{in(\psi-\pi)} = 0, \qquad \psi \in S' = (\beta \leq |\psi| \leq \pi) \tag{18b}$$

$$\sum_{(n)} \mu_n = 0 \tag{18c}$$

By introducing functions $X^{\pm}(z)$ of complex variable $z = re^{i\psi}$ such that

$$X^{\pm}(z) = \pm \sum_{n>0, n<0} \mu_n n (-1)^n z^n \tag{19}$$

we come to dual funtional equations

$$X^+(e^{i\psi}) + X^-(e^{i\psi}) = F'(e^{i\psi}), \qquad \psi \in M' \tag{20a}$$

$$X^+(e^{i\psi}) - X^-(e^{i\psi}) = 0, \qquad \psi \in S' \tag{20b}$$

which constitute an RHP of the type considered above with a curve $L = M' \bigcup S'$ being a unit circle L^0. If $X(z)$, like before, is vanishing at infinity and has singularities of the order of 1/2 at the endpoints $z = e^{\pm i\beta}$, then the solution is given by the equation (16).

However, we are interested in obtaining the coefficients μ_n describing the function $X(z)$ exactly on the curve L. So, we must make use of Plemelj-Sokhotskii formulas for the limiting values of $X(z)$ and find for $t, t_0 \in L$

$$X^+(t_0) - X^-(t_0) = \frac{Q(t_0)}{i\pi} \int_{M'} \frac{R^+(t) F'(t) dt}{t - t_0} + 2CQ(t_0) \tag{21}$$

where

$$Q(t_0) = \begin{cases} 1/R^+(t_0), & t_0 \in M' \\ 0, & t_0 \in S' = L^0 \backslash M' \end{cases}$$

As the definition (19) yields

$$X^+(t_0) - X^-(t_0) = \sum_{(n)} \mu_n n(-1)^n e^{in\psi_0}$$

we can make a Fourier inversion of (21) and obtain

$$\mu_m m(-1)^m = V_m(F', \beta) + 2CR_m(\beta), \qquad m = 0, \pm 1, \ldots \tag{22}$$

where we denoted

$$V_m(F,\beta) = \frac{1}{2\pi}\int_0^{2\pi} Q(e^{i\psi_0}) V(F, e^{i\psi_0}) e^{-im\psi_0} d\psi_0 = \frac{1}{2\pi}\int_{M'} \frac{V(F, e^{i\psi_0}) e^{-im\psi_0} d\psi_0}{R^+(e^{i\psi_0})}$$

$$V(F, e^{i\psi_0}) = \frac{1}{i\pi} P.V. \int_{M'} \frac{R^+(t) F(t) dt}{t - e^{i\psi_0}}$$

$$R_m(\beta) = \frac{1}{2\pi}\int_0^{2\pi} Q(e^{i\psi_0}) e^{-im\psi_0} d\psi_0 = \frac{1}{2\pi}\int_{M'} \frac{e^{-im\psi_0}}{R^+(e^{-i\psi_0})} d\psi_0$$

Setting $m = 0$ in (22), we can exclude the constant C, so that

$$\mu_m m(-1)^m = V_m - V_0 R_m / R_0 \tag{23}$$

Now, assume that the Fourier expansion of the free term function is available, that is

$$F'(e^{i\psi}) = \sum_{(n)} f_n e^{i\psi} \tag{24}$$

Then, obviously

$$V_m(F', \beta) = \sum_{(n)} f_n V_m^n$$

where

$$V_m^n = \frac{1}{2\pi}\int_{M'} \frac{v_n(e^{i\psi_0}, \beta)}{R^+(e^{i\psi_0})} e^{-im\psi_0} d\psi_0 \tag{25}$$

$$v_n(t_0, \beta) = \frac{1}{i\pi} P.V. \int_{M'} \frac{R^+(t) t^n}{t - t_0} dt \tag{26}$$

and (23) is reduced to

$$\mu_m m(-1)^m = \sum_{(n)} f_n (V_m^n - V_0^n R_m / R_0) \equiv \sum_{(n)} f_n \tilde{V}_m^n$$

Besides, we have to add the equation following from (18c) to find the coefficient μ_0:

$$\mu_0 = -\sum_{(n)} f_n \sum_{(m\neq 0)} (-1)^m \frac{\tilde{V}_m^n}{m} \tag{27}$$

Calculation of v_n, V_m^n and R_m requires a fine technique of integration in the complex plane and was performed by Agranovich et al.[6]. In terms of Legendre polynomials $P_n(\cos\beta)$ the results are as follows

$$\begin{aligned} V_m^n(\cos\beta) &= \frac{m+1}{2(m-n)}\left[P_m(\cos\beta)P_{n+1}(\cos\beta) - P_{m+1}(\cos\beta)P_n(\cos\beta)\right] \\ R_m(\cos\beta) &= \frac{1}{2}P_m(\cos\beta), \qquad \tilde{V}_m^n = V_{m-1}^{n-1} \\ \sum_{(m\neq 0)} (-1)^m \frac{V_{m-1}^{n-1}(\cos\beta)}{m} &= \begin{cases} -\frac{1}{n}V_{n-1}^{-1}(\cos\beta), & n\neq 0 \\ \ln\frac{1+\cos\beta}{2}, & n=0 \end{cases} \end{aligned} \tag{28}$$

So, we can rewrite the obtained equations in the final form convenient for further analysis

$$\mu_m = (-1)^m \sum_{(n)} f_n T_{mn}(\cos\beta) \tag{29}$$

where

$$T_{mn}(\cos\beta) = \begin{cases} \frac{1}{m}V_{m-1}^{n-1}(\cos\beta), & m\neq 0 \\ \frac{1}{n}V_{n-1}^{-1}(\cos\beta), & m=0, n\neq 0 \\ -\ln\frac{1+\cos\beta}{2}, & m=n=0. \end{cases} \tag{30}$$

3. FREE-SPACE H-WAVE SCATTERING BY AN OPEN CIRCULAR SCREEN

In this section, we treat the scattering of a plane H-polarized electromagnetic wave by an infinitely long circular cylinder with a longitudinal slot, and develop the solution in detail. As it was pointed out in Section 1.1, this problem was solved using the RHP technique by Koshparionok [14] (however, Sologub et al. [11] had obtained a more general solution for N-slitted cylinder somewhat earlier). Later, based on this formal solution, Nosich [15] and Ziolkowski [32] investigated numerically the resonances and far fields as well as near fields and surface currents, respectively.

All field quantities are assumed to have time variation $exp(-i\omega t)$with ω for the angular frequency, and this time factor will be omitted throughout all further sections.

3.1 Formulation of the Problem

Consider the scattering of a plane wave incident normally upon zero-thickness perfectly conducting circular nonclosed cylinder (screen), as shown in Fig. 2. The magnetic field vector of the incident wave is assumed to be parallel to the axis z of the cylindrical coordinate system (r, φ, z) coaxial with the scatterer. The screen's parameters are : radius a, angular

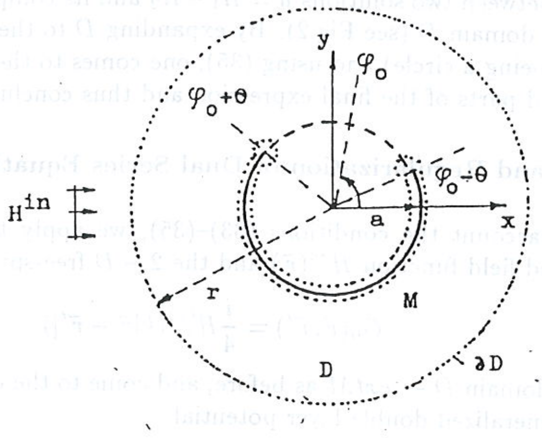

Fig. 2. On the plane-wave scattering from circular open screen in free space.

width 2δ, and those of the slot are: angular width $2\theta = 2(\pi - \delta)$ and position angle φ_0. Our aim is to reduce the scattering problem to dual series equations, and to solve (regularize) them by means of the RHP technique.

As the problem is essentially scalar and $2-D$ one, total field can be characterized by the single H_z component presented by the sum of incident and scattered field terms

$$H(\vec{r}) = H^{in}(\vec{r}) + H^{sc}(\vec{r}) = e^{ikx} + H^{sc}(\vec{r}), \qquad \vec{r} = (r, \varphi) \tag{31}$$

The function $H^{sc}(\vec{r})$ has to satisfy the $2-D$ Helmholtz equation

$$(\Delta + k^2)H^{sc}(\vec{r}) = 0, \qquad k = \omega/c \tag{32}$$

everywhere off the contour M of the screen, and the boundary condition on M of the Neumann type

$$\frac{\partial}{\partial n}(H^{in} + H^{sc}) = 0, \qquad \vec{r} \in M \tag{33}$$

Besides, due to the sharp edges of the screen, we need to impose a kind of restriction on the field behavior, namely, we demand that field energy is limited inside any bounded domain B

$$\int_B (k^2|H^{sc}|^2 + |\nabla H^{sc}|^2)d\vec{r} < \infty \tag{34}$$

including the vicinity of the edge.

Finally, if $Imk = 0$, far from the scatterer the well known Sommerfeld condition of radiation is demanded

$$H^{sc}(\vec{r}) \underset{r\to\infty}{\sim} \Phi(\varphi)\left(\frac{2}{i\pi kr}\right)^{1/2} e^{ikr} \tag{35}$$

It can be proved that the solution of the problem (31)–(35) is unique. To obtain this proof, one has to assume that opposite is true and apply the Green's formula

$$\int_D (u\Delta v - v\Delta u)d\vec{r} = \int_{\partial D} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) d\vec{r} \tag{36}$$

to the difference between two solutions $u = H_1 - H_2$ and its complex conjugate $v = H_1^* - H_2^*$ within a bounded domain D (see Fig.2). By expanding D to the whole space exterior to M (outer boundary being a circle) and using (35), one comes to the contradiction between the left and right hand parts of the final expression and thus concludes that $H_1 \equiv H_2$.

3.2 Derivation and Regularization of Dual Series Equations

Taking into account the conditions (33)-(35), we apply the Green's formula to the unknown scattered field function $H^{sc}(\vec{r})$ and the $2-D$ free-space Green's function

$$G_0(\vec{r}, \vec{r}') = \frac{i}{4} H_0^{(1)}(k|\vec{r} - \vec{r}'|) \tag{37}$$

within the same domain $D \longrightarrow extM$ as before, and come to the conclusion that solution can be sought as a generalized double-layer potential

$$H^{sc}(\vec{r}) = \int_M \mu(\vec{r}') \frac{\partial}{\partial n'} G_0(\vec{r}, \vec{r}') d\vec{r}' \tag{38}$$

where

$$\mu(\vec{r}') = [H(\vec{r}')], \qquad \vec{r}' \in M \tag{39}$$

is the unknown current density function (square brackets are for the jump of field function), and n is the exterior unit normal vector.

Now we can enforce the boundary condition (33) to the function (38) that yields an integro-differential equation

$$\frac{\partial}{\partial n} \int_M \mu(\vec{r}') \frac{\partial}{\partial n'} G_0(\vec{r}, \vec{r}') d\vec{r}' = -\frac{\partial}{\partial n} H^{in}(\vec{r}), \qquad \vec{r} \in M \tag{40}$$

So far we have not yet exploited the fact that the contour M is a part of a circle. If we take this into account, we can easily discretize our problem by expanding all the functions in (40) in terms of angular exponents $\exp(in\varphi)$, $n = 0, \pm 1, \ldots$.

Indeed, on completing the current density function with identical zero at the slot S, we expand it as follows

$$\mu(\vec{r}) = \frac{2}{i\pi ka} \sum_{(n)} \mu_n e^{in\varphi} \tag{41}$$

and immediately obtain a series equation for unknown coefficients

$$\sum_{(n)} \mu_n e^{in\varphi} = 0, \qquad \varphi \in S \tag{42}$$

Besides, by employing the well known addition theorem for cylindrical functions, we have

$$G_0(\vec{r}, \vec{r}') = \frac{i}{4} \sum_{(n)} \left\{ \begin{array}{ll} J_n(kr') H_n^{(1)}(kr), & r > r' \\ J_n(kr) H_n^{(1)}(kr'), & r < r' \end{array} \right\} e^{in(\varphi - \varphi')} \tag{43}$$

After substituting the above series to (40), term-by-term integration and differentiation (provided that functions are smooth enough) bring us another series equation valid on the complementary curve M

$$\sum_{(n)} \mu_n J_n'(ka) H_n^{(1)\prime}(ka) e^{in\varphi} = -\sum_{(n)} b_n e^{in\varphi}, \qquad \varphi \in M \tag{44}$$

with

$$b_n = i^n J_n'(ka) \tag{45}$$

Here we exploited the known expansion of the exponent, that is

$$e^{ikx} \equiv e^{ikr\cos\varphi} = \sum_{(n)} i^n J_n(kr) e^{in\varphi} \tag{46}$$

Thus, equations (42) and (44) constitute the dual series problem for a vector of coefficients $\{\mu\}_{n=-\infty}^{\infty}$ having a kernel of "trigonometric" type. It is equivalent to the initial boundary-value problem if the function $\mu(\varphi)$ is smooth enough to justify term-by-term differentiation in (44). Besides, we derive an additional requirement specifying the functional class of the sequence of μ_n, by applying the condition (34), say, to the domain $r \le a$, namely

$$\sum_{(n)} |\mu_n|^2 |n+1| < \infty \tag{47}$$

Now we have to reduce the obtained dual series equations to those of canonical form. Investigating asymptotic behavior of the product of cylindrical functions derivatives as $|n| \to \infty$, we find that

$$J_n'(ka) H_n^{(1)'}(ka) = -(i\pi k^2 a^2)^{-1} |n| \left[1 + O(k^2 a^2 n^{-2})\right] \tag{48}$$

Therefore, after introducting the set of functions

$$\Delta_n(ka) = |n| + i\pi(ka)^2 J_n'(ka) H_n^{(1)'}(ka) \tag{49}$$

we find that the following equations are valid

$$\sum_{(n)} \mu_n |n| e^{in\varphi} = \sum_{(n)} \left[\Delta_n \mu_n - i\pi(ka)^2 b_n\right] e^{in\varphi}, \qquad \theta < |\varphi - \varphi_0| \le \pi \tag{50a}$$

$$\sum_{(n)} \mu_n e^{in\varphi} = 0, \qquad |\varphi - \varphi_0| < \theta \tag{50b}$$

We see that the left hand parts of (50a,b) have exactly the same form as the equations investigated in Section 2.3. Thus, applying the RHP scheme as described above, we come to the infinite system of linear algebraic equations

$$\mu_m = \sum_{(n)} A_{mn} \mu_n + B_m, \qquad m = 0, \pm 1, \ldots \tag{51}$$

where

$$A_{mn} = \Delta_n \tilde{T}_{mn}(\theta, \varphi_0) \tag{52}$$

$$B_m = i\pi(ka)^2 \sum_{(n)} b_n \tilde{T}_{mn}(\theta, \varphi_0) \tag{53}$$

$$\tilde{T}_{mn}(\theta, \varphi_0) = (-1)^{m+n} e^{i(n-m)\varphi_0} T_{mn}(-\cos\theta) \tag{54}$$

with $T_{mn}(\cdot)$ given by (30).

This infinite system can be rewritten as a single operator equation

$$(I-A)\mu = B \tag{55}$$

where $\mu = \{\mu_n\}_{n=-\infty}^{\infty}$, operator I is that of identity and operator $A = ||A_{mn}||_{m,n=-\infty}^{\infty}$ can be shown to be compact in the Hilbert space of l_2: $\mu \in l_2$ if $\sum_{(n)} |\mu_n|^2 < \infty$. Actually, a more rapid convergence takes place as even $\sum_{(m,n)} |A_{mn}| < \infty$ due to the estimates

$$|A_{mn}| < \frac{c}{|m-n||m|^{1/2}|n|^{3/2}}, \qquad |A_{nn}| < \frac{c}{n^2} \tag{56}$$

which are established based on the known large-index behavior of cylindrical functions and Legendre polynomials.

It means that (55) is a regularized operator equation in l_2 with a canonical Fredholm operator, and well known Fredholm's theorems are valid : solution μ does exist for every real value of k and is unique. Further, this solution can be approximated with any desired accuracy by means of truncation of matrix $A = ||A_{mn}||_{m,n=-\infty}^{\infty}$ and vector $\{B_m\}_{m=-\infty}^{\infty}$ for all $|m|, |n| > N_{tr}$ because the sequence of approximate solutions is guaranteed to converge to exact one as $N_{tr} \to \infty$. In practice, a simple numerical rule has been verified : to provide the accuracy of 0.1%, one has to take N_{tr} =(entire part of) $ka + 5$. Finally, estimates (56) enable one to check that condition (34) is really satisfied as $|\mu_n| = O(|n|^{-3/2})$ with $|n| \to \infty$.

So, however (51) gives a formal (i.e. not analytical) solution of the considered problem, it can be treated as a numerically rigorous one due to being fully grounded from mathematical point of view.

3.3 Calculation of Near Field and Surface Current

According to (38), (41), (43) the field function is found to be

$$H(\vec{r}) = H^{in}(\vec{r}) + \sum_{(n)} \mu_n \Lambda_n(kr, ka) e^{in\varphi} \tag{57}$$

where

$$\Lambda_n(kr, ka) = \begin{cases} J_n'(ka) H_n^{(1)}(kr), & r \geq a \\ H_n'^{(1)}(ka) J_n(kr), & r \leq a \end{cases} \tag{58}$$

Large-index asymptotics of cylindrical functions enable one to see that as $|n| \to \infty$

$$\Lambda_n(kr, ka) \sim \frac{i}{\pi ka} \begin{cases} -(a/r)^n, & r \geq a \\ (r/a)^n, & r \leq a \end{cases} \tag{59}$$

Thus, the worst convergence is observed for $r = a$, and the same difficulty takes place for the calculation of surface current density function

$$j_\varphi(\varphi) = \frac{c}{4\pi}\left[H(a+0,\varphi) - H(a-0,\varphi)\right] = \frac{c}{4\pi}\mu(\varphi) \tag{60}$$

from the Fourier series (41) converging as $O(|n|^{-3/2})$.

To improve the convergence, one can extract large-n asymptotics of μ_n and summarize them explicitly (see [32]), but there is an alternative way. Let us substitute coefficients μ_n taken from (51) into (41) and interchange the order of summation. This yields

$$\mu(\varphi) = \frac{2}{i\pi ka} \sum_{(n)} \gamma_n (-1)^n e^{in\varphi_0} S_n(\theta, \varphi - \varphi_0) \tag{61}$$

where

$$\gamma_n = \Delta_n \mu_n + i\pi(ka)^2 b_n \tag{62}$$

$$S_n(\theta, \varphi) = \sum_{(m)} (-1)^m e^{im\varphi} T_{mn}(-\cos\theta) \tag{63}$$

The sums like (63) can be obtained analytically. Indeed,

$$S_n(\theta, \varphi) = -i \int_{\pi}^{-\varphi'} \sum_{(m\neq 0)} V_{m-1}^{n-1}(u') e^{-im\beta} d\beta \tag{64}$$

where we defined $\varphi' = \pi - \varphi$, $u' = -\cos\theta = \cos\theta'$, $\theta' = \theta - \pi$

Then we have to make use of integral expression (25) for $V_{m-1}^{n-1}(u')$ and representation (26) for $v_n(t_0, \beta)$, and take the integrals

$$I_n(u) = \int t^n R^{-1/2}(t, u) dt \tag{65}$$

where $R(t, u) = t^2 - 2ut + 1$, obtaining a set of recurrent expressions

$$\begin{aligned} I_0 &= \ln(2R^{1/2} + 2t - 2u) \\ I_1 &= R^{1/2} + uI_0 \\ nI_n &= t^{n-1} R^{1/2} + (2n-1)uI_{n-1} - (n-1)I_{n-2} \end{aligned} \tag{66}$$

After integrating, the final result is as follows

$$\begin{aligned} S_0(\theta, \varphi) &= U(\varphi) \ln \left[\frac{1 + \cos\varphi'}{1 + u'} \left(1 + \left\{ 1 - \frac{1 + u'}{1 + \cos\varphi'} \right\}^{1/2} \right)^2 \right] \\ S_n(\theta, \varphi) &= U(\varphi)(2\cos\varphi' - 2u')^{1/2} e^{i\varphi'/2} \sum_{k=0}^{n} C_k Q_{n-k}(e^{i\varphi'}) \end{aligned} \tag{67}$$

where $U(\varphi) = 0$ if $\varphi \in S$ or 1 if $\varphi \in M$

$$\begin{aligned} C_0 &= 1 \\ C_1 &= -u' \\ C_k &= P_k(u') - 2u' P_{k-1}(u') + P_{k-2}(u') \end{aligned} \tag{68}$$

$$\begin{aligned} Q_1(x) &= 1 \\ Q_2(x) &= (x + 3u')/2 \\ nQ_n(x) &= x^{n-1} + (2n-1)u' Q_{n-1}(x) - (n-1)Q_{n-2}(x) \end{aligned} \tag{69}$$

Thus, the resulting expressions for $S_n(\theta,\varphi)$ are products of smooth functions, step function $U(\varphi)$, and a square root term of the precisely same form as needed by the edge condition (34). The new series (61) now converges as $O(|n|^{-3})$ and besides it satisfies the edge condition in term-by-term manner and independently of the accuracy of finding μ_n. A similar procedure can be used to enhance convergence of the series expressions for any other field component at the circle $r=a$. These derivations show that all the operations preceding (51) are really valid and the latter are equivalent to the initial problem.

3.4 Far-Field Scattering Characteristics

As for the far-field scattering characteristics, the field pattern introduced in (35) is found from (57) to be

$$\Phi(\varphi)=\sum_{(n)}\mu_n(-i)^n J_n'(ka)e^{in\varphi} \tag{70}$$

Besides, the total and backward scattering cross-sections (TSC and BSC) give the amount of total scattered power and that of the fracture reflected back to the source, respectively. In terms of μ_n coefficients they are

$$\sigma_s=\frac{2}{\pi k}\int_0^{2\pi}|\Phi(\varphi)|^2 d\varphi=\frac{4}{k}\sum_{(n)}|\mu_n J_n'(ka)|^2 \tag{71}$$

$$\sigma_b=\frac{4}{k}|\Phi(\pi)|^2=\frac{4}{k}|\sum_{(n)}\mu_n i^n J_n'(ka)|^2 \tag{72}$$

In Figs. 3, 4, the typical frequency dependences of TSC and BSC are shown. In the event of narrow slit (Fig.3), sharp resonances are observable due to the excitation of the damped natural modes of the screen as a cavity-backed aperture. Exept the first, low-frequency, peak all these modes originate from the eigenmodes of circular cavity, being shifted in frequency (and splitted into pairs $H_{mn}^{\pm}$ for $m\neq 0$) by cutting the slot. As for the low-frequency resonance, it has a singular nature. Formally, any interior Neumann boundary-value problem has the zero as the lowest eigenvalue but in electromagnetics the corresponding eigenfunction happens to be identically zero everywhere. Cutting the slot shifts this zero value by a small complex number, and the corresponding (generalized) eigenfunction is now not equal to zero. This resonance is of the same nature as so-called Helmholtz mode of oscillations in acoustic cavities investigated by Lord Rayleigh [35]. We denote it as H_{00}, and address [15] to further details.

Note that all the peaks of TSC and BSC are of finite amplitude because all the perturbed eigenfrequencies are now complex numbers with negative imaginary parts found to be proportional either to $O(\ln^{-1}\theta)$ for H_{mn}^{+} modes or to $O(\theta^2)$ for H_{mn}^{-} modes ($m\neq 0$) [15]. Helmholtz resonance is of a special interest as its eigenfrequency is

$$k_{00}\approx a^{-1}\left(-2\ln\sin\frac{\theta}{2}\right)^{-1/2}\left(1+\frac{i\pi}{16}\ln^{-1}\sin\frac{\theta}{2}\right) \tag{73}$$

tending to zero as $\theta\to 0$. It means that the Helmholtz mode destroys the well known Rayleigh rule for the H-scattering by small objects according to which TSC and BSC are proportional to $O(k^3a^4)$. Actually, at the resonant frequency k_{00}, we have

$$\sigma_s=\sigma_b=4/k_{00}+O(k_{00}a^2) \tag{74}$$

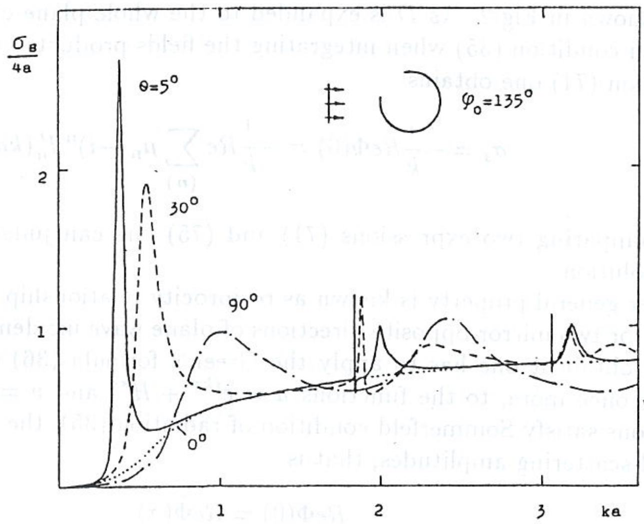

Fig. 3. Normalized total scattering cross-sections versus frequency parameter *ka* for closed circular bar and open screens with various apertures.

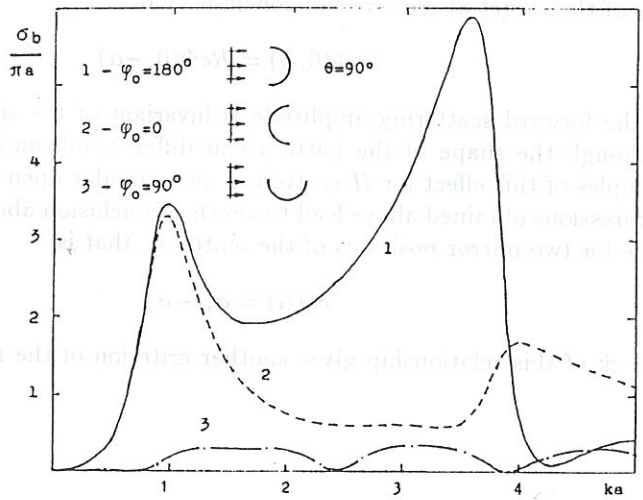

Fig. 4. Frequency dependences of normalized backscattering cross-sections for three space positions of semicircular screen.

for a slitted cylinder (see also Section 3.5) with an almost isotropic scattering pattern $\Phi(\varphi)$. From a physical point of view, at the exitation of H^{+}_{mn} mode the slot is radiating as an intensive secondary line source (as a pair of opposite-phased sources for H^{-}_{mn} mode), and for H_{00} mode the effect of cylinder on this radiation is very small. As the slot is widened, Helmholtz-mode resonance transfers to the $\lambda/2$ resonance of a curved strip.

There exist certain general properties of the scattered field valid for arbitrary shaped scatterer. One of these features is formalized by the energy conservation equation (or so-called Optical Theorem). To derive it, one has to apply Green's formula (36) to the total field function and its complex conjugate (i.e. take $u = H = e^{ikx} + H^{sc}$, $v = H^{*}$) in a domain D shown in Fig.2. As D is expanded to the whole plane exterior to M, one has to use radiation condition (35) when integrating the fields products on ∂D. Taking account of TSC definition (71) one obtains

$$\sigma_s = -\frac{4}{k} Re\Phi(0) = -\frac{4}{k} Re \sum_{(n)} \mu_n (-i)^n J_n'(ka) \tag{75}$$

Thus, by comparing two expressions (71) and (75) one can judge about the accuracy of numerical solution.

Another general property is known as reciprocity relationship which couples scattering amplitudes for two mirror-opposite directions of plane wave incidence on the same scatterer. In order to obtain it one has to apply the Green's formula (36) within the same domain $D \to extM$ once more, to the functions $u = e^{ikx} + H^{sc}$ and $v = e^{-ikx} + \tilde{H}^{sc}$. Again, as both functions satisfy Sommerfeld condition of radiation (35), the final result contains only the forward scattering amplitudes, that is

$$Re\Phi(0) = Re\tilde{\Phi}(\pi) \tag{76}$$

If a $2-D$ scatterer has a plane of symmetry (as an open circular screen), its position can be characterized through the angle α between this plane and y-axis. Taking into account that two mirror directions of plane wave incidence along x-axis are equivalent to two mirror orientations of the target at $\pm\alpha$, we can conclude that

$$Re\Phi(0, \alpha) = Re\Phi(0, -\alpha) \tag{77}$$

Thus, the forward scattering amplitude is invariant of the sign of the target position angle α although the shape of the pattern can differ significantly. Fig. 5 demonstrates several examples of this effect for H-scattering by a circular open screen.

The expressions obtained above lead to another conclusion about the TSC values : they are the same for two mirror positions of the scatterer, that is

$$\sigma_s(\alpha) = \sigma_s(-\alpha) \tag{78}$$

The check of this relationship gives another criterion of the accuracy of numerical solution.

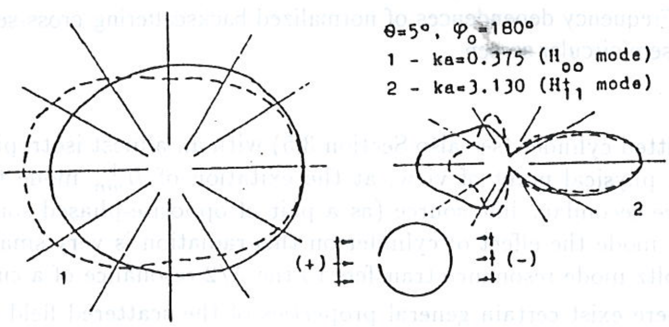

Fig. 5. Far-field scattering patterns at resonant frequencies for a narrow-slitted circular screen.

3.5 Low-Frequency Asymptotic Solution

Although the solution of the studied problem is available numerically for arbitrary set of parameters, equations (51) are also suitable for extracting some valuable results analytically in the form of asymptotic series. Indeed, exloiting small-argument approximations for cylindrical functions in the matrix elements, we find that

$$q = \max_{m} \sum_{(n)} |A_{mn}| < c(1+u)(ka)^2 \tag{79}$$

where $u = \cos\theta$.

It follows from (79) that for either $ka \to 0$ or $\delta = \pi - \theta \to 0$ (narrow strip case) we can use a convergent iterative operator series

$$\mu = \sum_{s=0}^{\infty} A^s B \tag{80}$$

Similar but somewhat different iterative technique can be used for $\theta \to 0$ (narrow slit case), as

$$\tilde{q} = \max_{m \neq 0} \{(1 - A_{mm})^{-1} \sum_{(n \neq 0)} |A_{mn}|\} < c(1-u^2)(ka)^2 \tag{81}$$

This approach yeilds the following low-frequency asymptotics for the scattering pattern, TSC and BSC ($\kappa = ka \to 0$)

$$\Phi(\varphi) \approx -\frac{1}{2}\kappa\mu_0 + \frac{i}{2}\kappa^2(1+u)\Big[2\mu_0\cos(\varphi - \varphi_0) + \pi\cos\varphi + \frac{\pi}{2}(1-u)\cos(\varphi - 2\varphi_0)\Big] \tag{82}$$

where

$$\mu_0 \approx \frac{i\pi\kappa^2}{2D(\kappa,\theta)}\Big[\kappa\frac{i(1+u)}{T_{00}}\cos\varphi_0 - \kappa\frac{1-2u-3u^2}{8T_{00}}\cos 2\varphi_0\Big] \tag{83}$$

$$D(\kappa,\theta) \approx \frac{1}{T_{00}} - \kappa^2\left(1 + \frac{i\pi\kappa^2}{4}\right) \tag{84}$$

$$T_{00} = T_{00}(-\cos\theta) = -\ln\sin^2\frac{\theta}{2}$$

and

$$\sigma_s \approx \frac{\kappa^2}{k}\Big\{|\mu_0|^2\left(1 + 2\kappa^2\cos^4\frac{\theta}{2}\right) + \kappa^2\cos^4\frac{\theta}{2} \Big[2\pi Re\mu_0\left(1+\sin^2\frac{\theta}{2}\right)\cos\varphi_0 + \frac{\pi^2}{2}\left(1+\sin\frac{\theta}{2}\right) + \pi^2\sin^2\frac{\theta}{2}\cos 2\varphi_0\Big]\Big\} \tag{85}$$

$$\sigma_b \approx \frac{\kappa^2}{k}\Big\{|\mu_0|^2\left(1 + 4\kappa^2\cos^4\frac{\theta}{2}\cos^2\varphi_0\right) + 2\,\mathrm{Im}\,\mu_0 + 4\kappa Re\mu_0\cos^2\frac{\theta}{2}\cos\varphi_0 + \pi\kappa\cos^2\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2}\cos 2\varphi_0\right)\Big\} \tag{86}$$

These formulas give asymptotic solutions uniform with respect to the parameters θ and φ_0. They are valid for a small open circular screen of arbitrary width, from an arbitrary positioned narrow strip to a circular cylinder, and describe also the resonant responce at the Helmholtz mode frequency ($\kappa \approx k_{00}a$). The main term of TSC and BSC at this situation is given by (74). Note that this term does not depend on the slot position angle φ_0. Indeed, as in H-scattering the induced current flows across the scatterer, there is no actual shadow region on its surface, and the cavity excitation is equally efficient for any position of aperture.

The above analysis has been restricted to the case of H-polarized scattering. However, the alternative case of E-polarization is treatable by the RHP technique as well [14]. The diffraction of plane E-wave by an open circular screen was studied numerically in [36]. For this polarization the only induced surface current component is that parallel to the axis of the screen, and this fact accounts for somewhat different behavior of the field due to much more pronounced effect of shadowing.

In the papers [14–19, 22, 23] one can find a vast amount of numerical results on the H-wave scattering by finite number of open circular screens modelling various microwave devices, and by infinite periodic grating of such screens.

4. SCATTERING BY SCREENS IN STRATIFIED DIELECTRIC MEDIUM

This section deals with the problems of scattering of plane or guided waves from same open screens as in previous section but placed in a stratified plane-parallel dielectric medium. It is shown that these problems are also treatable with the RHP technique that is completed with the Green's function approach to take into account properly the material and geometric parameters of the medium. As examples, H-scattering from screens near a plane boundary between two dielectrics, near a plane impedance surface, and inside a parallel-plate dielectric slab is analyzed, and both analytical and numerical results are presented. Perhaps, among the most interesting are the problems simulating the guided modes scattering from inhomogeneities. Similar problems on impedance-plane surface wave scattering from circular cylinders have been treated approximately by Davies and Leppington [37], and on dielectric-slab modes scattering by Morita [38], Uzunoglu and Fikioris [39], and Kalinichev et al. [40,41]. As for the scattering from a circular screen, the solution was first given by Nosich for a screen near impedance plane [42], and later for arbitrary collection of screens in stratified medium [43]. Some numerical results about plane and cylindrical wave diffraction can be found in [44] and about slab modes scattering and conversion in [45].

4.1 On the Modified Condition of Radiation

The most general scattering geometry which is to be considered in this section is shown in Fig.6. There is a material slab D_ε consisting of M_ε parallel layers characterized with dielectric constants $\varepsilon_j (j = 1, \ldots M_\varepsilon)$, sandwiched between two halfspaces with dielectric constants $\varepsilon_\pm$. All the scatteres, namely N open screens, are contained within a finite region. Incident field is specified by some excitation field function $H^{in}(\vec{r})$. The boundary-value problems associated with this geometry are not "classical" diffraction problems in open domain in the sense that here some of the surfaces, at which the boundary conditions are given, are not finite. It means that although we need some condition of infinity ($r \to \infty$) to complete the formulation, the conventional Sommerfeld one is not necessary valid for a lossless problem ($\mathrm{Im}\,\varepsilon_j = O,\ \forall_j$), as it was derived for compact scatterers with finite boundaries. Thorough investigation of this question was carried out by Nosich [34] who

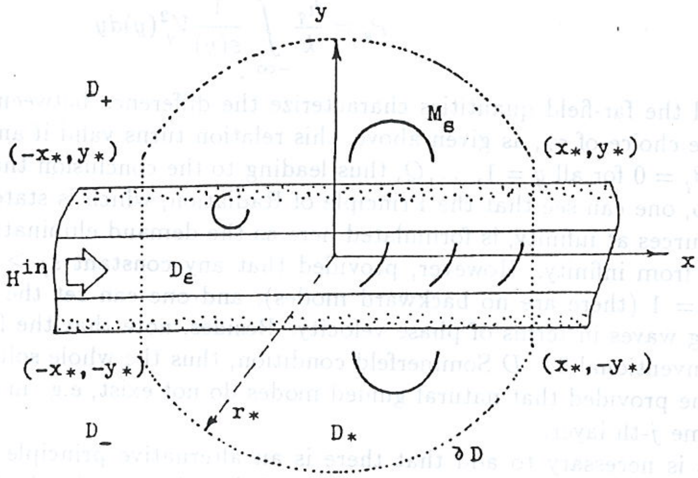

Fig. 6. On the scattering from multiple screens in stratified dielectric medium.

showed that a more general condition ensures the uniqueness of the solution, and is stated as

$$H^{sc}(\vec{r}) \underset{r\to\infty}{\simeq} \begin{cases} \Phi^{\pm}(\varphi)\left(\frac{2}{i\pi k_{\pm} r}\right)^{1/2} e^{ik_{\pm} r}, & \vec{r} \in D_{\pm} \\ 0, & r \in D_{\varepsilon} \end{cases}$$
$$+ \sum_{q=1}^{Q} \begin{Bmatrix} \alpha_q, & x > 0 \\ \beta_q, & x < 0 \end{Bmatrix} V_q(y) e^{i\gamma_q h_q |x|} \tag{87}$$

where $[H_q \equiv] V_q(y) e^{ih_q x} (q = 1, \ldots, Q)$ are the natural guided (nondecaying) modes of the sandwich-dielectric slab, propagating with real wavenumbers h_q: $\min k_j < h_q < \max k_j$, $k_j = k\varepsilon_j^{1/2} (j = -, +, 1, \ldots, M_\varepsilon)$, while parameters $\gamma_q = sign(dh_q/dk) = \pm 1$ correspond to positive or negative group velocity of modes.

To prove the uniqueness one must act similarly to the free-space scattering case (see Section 3.1). However, to conform to the open-waveguide scattering geometry it is necessary to integrate with weight $\epsilon^{-1}(y)$ and use domain D_* of modified shape. This domain is now bounded by the parts of a circle of radius r_* upper and below the slab, and straight-line sections $(x = \pm x_*, |y| < y_*)$ shown in Fig.6. When expanding D_* to the whole plane, the passing to the limit is made in such a manner that $x_*, y_*, r_* \to \infty$ but $y_*/r_* \to 0$. Then integrating over the curved parts of ∂D_* one can take account of only the cylindrical-wave term of (87), while at the straight-line portions – only the guided-modes term. Exploiting well-known orthogonality properties of guided modes in the cross-section, one obtains

$$\frac{2}{\pi k} \int_0^{\pi} \left(|\Phi^+|^2 + |\Phi^-|^2\right) d\varphi + \sum_{q=1}^{Q} \gamma_q \left(|\alpha_q|^2 + |\beta_q|^2\right) P_q = 0 \tag{88}$$

where

$$P_q = \frac{h_q}{k} \int_{-\infty}^{\infty} \frac{1}{\varepsilon(y)} V_q^2(y) dy \qquad (89)$$

and all the far-field quantities characterize the difference between two solutions $H_1 - H_2$. For the choice of γ_q, as given above, this relation turns valid if and only if $\Phi^{\pm}(\varphi) \equiv 0$ and $\alpha_q = \beta_q = 0$ for all $q = 1, \ldots, Q$, thus leading to the conclusion that $H_1 \equiv H_2$.

So, one can see that the Principle of Radiation, which is stated as the one eliminating any sources at infinity, is formulated here as the demand eliminating any wave that carries power from infinity. However, provided that any constant $\varepsilon_j \geq 1$, it can be shown that all $\gamma_q = 1$ (there are no backward modes), and one can set the demand eliminating any ingoing waves in terms of phase velocity. Besides, note that the first term in (87) satisfies the conventional $2-D$ Sommerfeld condition, thus the whole solution can be extracted by this one provided that natural guided modes do not exist, e.g. in the event that $Im\varepsilon_j > 0$, for some j-th layer.

It is necessary to add that there is an alternative principle of extracting the unique solution of a lossless problem in unbounded domain, namely, the Principle of Limiting Absorption. That is, the solution is taken as

$$H(\vec{r}, k) = \lim_{\tilde{k} \to +0} \tilde{H}(\vec{r}, k + i\tilde{k}) \qquad (90)$$

where $H(\vec{r}, k + i\tilde{k})$ vanishes at infinity and is the solution of a lossy problem, i.e. the problem with losses assumed in the outer media. Unfortunately, there are no general theorems ensuring universal validity of this principle. Existence of a limit like (90) has to be proved independently for different classes of boundary-value problems. As for regular open waveguides, it has been shown in [34] that this principle is really valid here and extracts exactly the same solution as the modified condition of radiation (87). Besides, it turns out that the limit like (90) can be used if the losses are assumed inside some of the dielectric layers, say $\varepsilon_j' = \varepsilon_j + i\tilde{\varepsilon}_j$. Then for $\tilde{H}(\vec{r}, \varepsilon_j')$ one must take the solution of a lossy problem governed by the Sommerfeld condition of radiation like (35), as a lossy dielectric slab cannot support any nondecaying guided mode.

To conclude this remark about field behaviour at infinity, it is worth to point out that the quantities $\Phi^{\pm}(\varphi)$ and α_q, β_q $(q = 1, \ldots, Q)$ are the scattering pattern and mode excitation coefficients, respectively. These parameters as functions of frequency and geometry of scatterers are sought usually for practical analysis.

4.2 Screen Near a Plane Dielectric Interface

Consider the diffraction of an H-polarized plane wave by a perfectly conducting infinitely thin open circular screen near the infinite plane interface between two halfspaces of different dielectric properties, as shown in Fig.7. The upper and lower media are denoted as $D_{\pm} = (y > 0, y < 0)$ and dielectric permittivity is characterized by a piece-constant function $\varepsilon(\vec{r}) = 1$ for $\vec{r} \in D_+$ and $\varepsilon(\vec{r}) = \varepsilon$ for $\vec{r} \in D_-$.

Open screen with arbitrary parameters a, θ, φ_0 is located at the distance b above the interface, not intersecting it. Let the structure be excited by the plane H-wave incident at the angle $\beta (0 < \beta < \pi)$ with respect to the interface. For the sake of convenience, take the initial field as the one in the absence of the screen, that is

$$H^{in}(\vec{r}) = \begin{cases} e^{ikx\cos\beta}\left[e^{-iky\sin\beta} + \tilde{r}(\beta)e^{iky\sin\beta}\right], & y > 0 \\ \tilde{t}(\beta)e^{ik\varepsilon^{1/2}(x\cos\gamma - y\sin\gamma)}, & y < 0 \end{cases} \qquad (91)$$

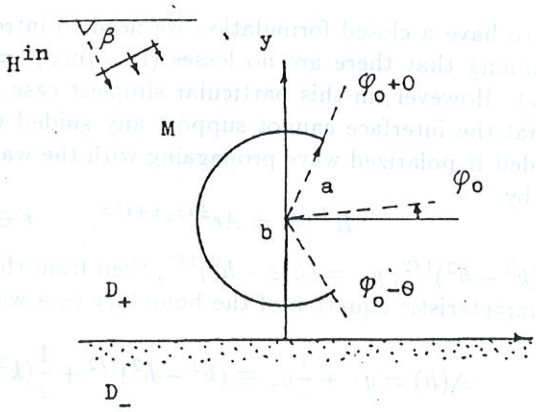

Fig. 7. Scattering geometry for a screen above plane dielectric interface. Plane wave is incident.

Here complex quantities $\tilde{r}(\beta)$ and $\tilde{t}(\beta)$ are the reflection and transmission coefficients which can be easily found from the continuity conditions

$$[H^{in}] = 0, \quad \left[\frac{1}{\varepsilon}\frac{\partial}{\partial y} H^{in}\right] = 0, \quad y = 0 \tag{92}$$

where square brackets are for the jumps of corresponding functions, with the result

$$\tilde{r}(\beta) = (\sin\beta - \varepsilon^{-1/2}\sin\gamma)\tilde{\Delta}^{-1}(\beta) \tag{93}$$

$$\tilde{t}(\beta) = 2\sin\beta\tilde{\Delta}^{-1}(\beta) \tag{94}$$

$$\tilde{\Delta}(\beta) = \sin\beta + \varepsilon^{-1/2}\sin\gamma \tag{95}$$

Besides, the angle of incidence β and that of refraction γ are coupled by the Snell's Law

$$\cos\beta = \varepsilon^{-1/2}\cos\gamma \tag{96}$$

It is known that the equations (93)–(95) describe as well the plane wave incidence at the boundary between two dielectrics with arbitrary permittivities ε_+ and ε_-. In this event the quantity ε must be understood as the value of contrast : $\varepsilon = \varepsilon_-/\varepsilon_+$.

If we define the total field in the presence of the screen as before

$$H(\vec{r}) = H^{in}(\vec{r}) + H^{sc}(\vec{r}) \tag{97}$$

then the scattered field $H^{sc}(\vec{r})$ satisfies the problem similar to (32)–(34) but subject to the continuity conditions like (92) at the interface, that is

$$[\Delta + k^2\varepsilon(\vec{r})]\, H^{sc}(\vec{r}) = 0, \qquad \vec{r} \in R^2 \backslash (M, y = 0) \tag{98}$$

$$\frac{\partial}{\partial n}(H^{in} + H^{sc}) = 0, \qquad \vec{r} \in M \tag{99}$$

$$\int_B (k^2\varepsilon|H^{sc}|^2 + |\nabla H^{sc}|^2)\, d\vec{r} < \infty, \quad \forall B \subset R^2 \tag{100}$$

$$[H^{sc}] = 0, \quad \left[\frac{1}{\varepsilon}\frac{\partial}{\partial y} H^{sc}\right] = 0, \qquad y = 0. \tag{101}$$

Besides, to have a closed formulation we need to introduce the adequate condition at infinity. Assuming that there are no losses (i.e. $\mathrm{Im}\,\varepsilon_\pm = 0$), we come to an expression similar to (87). However, in this particular simplest case of a dielectric stratified medium it happens that the interface cannot support any guided wave. Indeed, if we assume that there is a guided H-polarized wave propagaing with the wavenumber h along x-axis, and its field is given by

$$H^\pm(\vec{r}) = Ae^{\pm ig_\pm y + ihx}, \qquad \vec{r} \in D_\pm \tag{102}$$

where $g_+ = (k^2 - h^2)^{1/2}$, $g_- = (k^2\varepsilon - h^2)^{1/2}$, then from the continuity conditions (101), we obtain the characteristic equation of the boundary as a waveguiding surface

$$\Delta(h) = g_+ + \frac{1}{\varepsilon}g_- \equiv (k^2 - h^2)^{1/2} + \frac{1}{\varepsilon}(k^2\varepsilon - h^2)^{1/2} = 0 \tag{103}$$

To have no attenuation, h must be real and both transverse wavenumbers g_+ and g_- are to be purely imaginary with positive imaginary parts, but as for dielectrics $\varepsilon > 0$, equation (103) demands that these values must be opposite. So, if $\varepsilon > 0$ characteristic equation has no such real roots that $k < |h| < k\varepsilon^{1/2}$. Thus, one concludes that in a general expression (87) the second term vanishes, and the field at infinity has to behave like

$$H^{sc}(\vec{r}) \underset{r\to\infty}{\sim} \Phi^\pm(\varphi)\left(\frac{2}{i\pi k_\pm r}\right)^{1/2} e^{ik_\pm r}, \qquad \vec{r} \in D_\pm. \tag{104}$$

Let us seek the solution of the problem (98)–(101), (104) as a generalized double-layer potential, formally coinciding with (38) where now we should use not $G_0(\vec{r}, \vec{r'})$ but $G(\vec{r}, \vec{r'})$, being the Green's function of 2-D space divided to two halfspaces $D_\pm$ by a plane boundary at $y = 0$. $G(\vec{r}, \vec{r'})$ has to satisfy the problem very similar to given above but without the conditions (99), (100) associated with the screen, and with a Dirac delta-function in the right hand part of (98).

From the most general point of view this function can be presented as the sum of singular and regular parts (assume that $\vec{r'} \in D_+$):

$$G(\vec{r}, \vec{r'}) = \begin{cases} G_0(\vec{r}, \vec{r'}) + G_+^{sc}(\vec{r}, \vec{r'}), & \vec{r} \in D_+ \\ G_-^{sc}(\vec{r}, \vec{r'}), & \vec{r} \in D_- \end{cases} \tag{105}$$

and the regular part can be treated as a Fourier-type integral

$$G_\pm^{sc}(\vec{r}, \vec{r'}) = \frac{i}{4\pi}\int_C \frac{1}{g_+}\begin{Bmatrix} r(h), D_+ \\ t(h), D_- \end{Bmatrix} e^{\pm ig_\pm y + ig_+ y' + ih(x-x')}dh \tag{106}$$

with the coefficients at Fourier transforms given by

$$r(h) = (g_+ - \varepsilon^{-1}g_-)\Delta^{-1}(h), \tag{107}$$

$$t(h) = 2g_+\Delta^{-1}(h), \tag{108}$$

where $\Delta(h)$ is defined by (103), so one can see that $r(k\cos\beta) = \tilde{r}(\beta)$, $t(k\cos\beta) = \tilde{t}(\beta)$, $\Delta(k\cos\beta) = k\tilde{\Delta}(\beta)$. The integration path C here is composed of semicircles centered at the singularities of the integrand and remaining portions of the real h-axis as it is shown in Fig.8. In all there are four branch-points $h = \pm k$, $h = \pm k\varepsilon^{1/2}$, so the proper sheet of the corresponding Riemann surface is extracted by conditions $Re g_\pm \geq 0$, $\mathrm{Im}\, g_\pm \geq 0$.

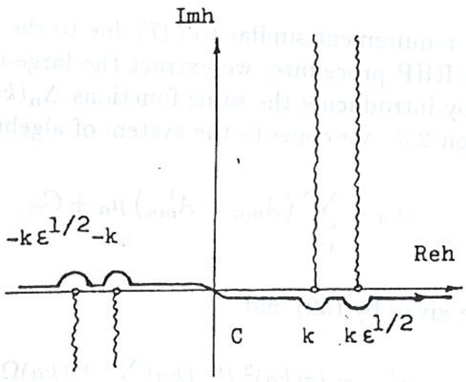

Fig. 8. Complex h-plane with branch points and initial integration path C.

Enforcing the boundary condition (99) on the screen's surface M, we derive an equation similar to (40). To reduce it to the dual series problem, we expand all the functions in terms of Fourier exponential series in the local coordinates of the screen $\vec{r}_c = (r_c, \varphi_c) = (x, y-b)$. That is, we use the series (41) for the current density function $\mu(\vec{r}_c)$ and the series like (46) for the incident field (91) that yields the following expansion coefficients for the normal derivative at M

$$b_n = i^n J_n'(ka)\left[e^{-ikb\sin\beta+in\beta} + \tilde{r}(\beta)e^{ikb\sin\beta-in\beta}\right] \tag{109}$$

As for the Green's function, discretizing $G_0(\vec{r}, \vec{r}')$ we use (43), and similarly

$$G_\pm^{sc}(\vec{r}_c, \vec{r}_c') = \frac{i}{4}\sum_{(n)}\sum_{(p)} J'_{-n}(k_+ r_c')J_p(k_\pm r_c)\Omega^\pm_{np}(kb,\varepsilon)e^{ip\varphi_c - in\varphi_c'} \tag{110}$$

where we introduce functions

$$\Omega^\pm_{np}(kb,\varepsilon) = \frac{i^{n+p}}{\pi}\int_C \frac{1}{g_+}\left\{\begin{matrix} r(h)e^{i2g_+b} \\ t(h)e^{i(g_+-g_-)b}\end{matrix}\right\} e^{in\psi_+ \pm ip\psi_-}\, dh \tag{111}$$

and define $\psi_\pm(h)$ by expressions

$$\cos\psi_\pm = h/k_\pm, \quad \sin\psi_\pm = -g_\pm/k_\pm = -(1-h^2/k_\pm^2)^{1/2}, \quad k_+ = k, k_- = k\varepsilon^{1/2} \tag{112}$$

Substituting these series into (40), we can integrate it term-by-term taking account of the orthogonality of exponents and make term-by-term differentiation assuming that this is possible. After setting $r_c = a$ we obtain the dual series equations

$$\sum_{(n)}\mu_n\left[J_n'(ka)H_n^{(1)\prime}(ka)e^{in\varphi_c} + J'_{-n}(ka)\sum_{(p)} J_p'(ka)\Omega^+_{np}(kb,\varepsilon)e^{ip\varphi_c}\right]$$
$$= -\sum_{(n)} b_n e^{in\varphi_c}, \qquad \varphi_c \in M \tag{113a}$$

$$\sum_{(n)}\mu_n e^{in\varphi_c} = 0, \qquad \varphi_c \in S \tag{113b}$$

with additional requirement similar to (47) due to the edge condition (100).

To use the RHP procedure, we extract the large-n behaviour term from the left hand part of (113a) by introducing the same functions $\Delta_n(ka)$ as before (see (49)). Applying the results of Section 2.3 , we come to the system of algebraic equations

$$\mu_m = \sum_{(n)} (A_{mn} + A^1_{mn})\, \mu_n + C_m, \qquad m = 0, \pm 1, \ldots \tag{114}$$

where A_{mn} are given by (52) and

$$A^1_{mn} = i\pi(ka)^2 J'_{-n}(ka) \sum_{(p)} J'_p(ka)\Omega^+_{np}(kb,\varepsilon)\tilde{T}_{mp}(\theta,\varphi_0) \tag{115}$$

$$C_m = i\pi(ka)^2 \sum_{(n)} b_n \tilde{T}_{mn}(\theta,\varphi_0) \tag{116}$$

with b_n given by (109) and $\tilde{T}_{mn}(\theta,\varphi_0)$ given by (54).

In operator form equations (114) can be rewritten as

$$(I - A - A^1)\mu = C \tag{117}$$

As we already know, the operator A is compact in l_2, and operator A^1 can be shown of the same type provided that the screen does not intersect the interface. So, for any $C \in l_2$ we conclude about the existence and uniqueness of the solution and also about the possibility of its approximation through truncation. A procedure similar to that from Section 3.3 enables one to check that the function $\mu(\vec{r'})$ calculated on the basis of (114) satisfies the edge condition and is smooth enough to ensure the validity of operations preceding (114).

A note should be made that in the event that the screen is located under the interface, we can obtain the equations similar to (114) with the quantities $k_+, \Omega^+_{np}(kb,\varepsilon)$ replaced with $k_-, \Omega^+_{np}(k_- b,\varepsilon)$ and free-term coefficients taken as $b_n = \tilde{t}(\beta) i^n J'_n(k_- a) e^{in\gamma + ik_- b\sin\gamma}$.

The far-field scattering pattern is now calculated through a more difficult procedure than in free-space case. Interchanging the order of integration along contour M in (38) and that in the Fourier representation of the Green's function (106) in D_+ yields

$$H^{sc}(\vec{r}) = \frac{ai}{4} \int_C \frac{e^{ihx_c}}{g_+} \int_M \mu(\vec{r}'_c) \frac{\partial}{\partial n'_c} \left[e^{-ihx'_c + ig_+|y_c - y'_c|} + r(h) e^{-ihx'_c + ig_+(y_c + y'_c + 2b)} \right] d\vec{r}'_c dh \tag{118}$$

Now $H^{sc}(\vec{r})$ can be evaluated at $r \to \infty$ following the asymptotic integration. Out of the region defined by the inequalities $|kx| >> 1$, $|kx| >> k^2(y + 2b)^2$ we can exploit the saddle-point technique (see [46] for details) that yields

$$\Phi^+(\varphi_c) = \frac{i}{4} \int_M \mu(\vec{r}'_c) \frac{\partial}{\partial n'_c} \left[e^{ik(y'_c \sin\varphi_c - x'_c \cos\varphi_c)} + \tilde{r}(\varphi_c) e^{i2kb\sin\varphi_c} e^{-ik(y'_c \sin\varphi_c + x'_c\cos\varphi_c)} \right] d\vec{r}'_c \tag{119}$$

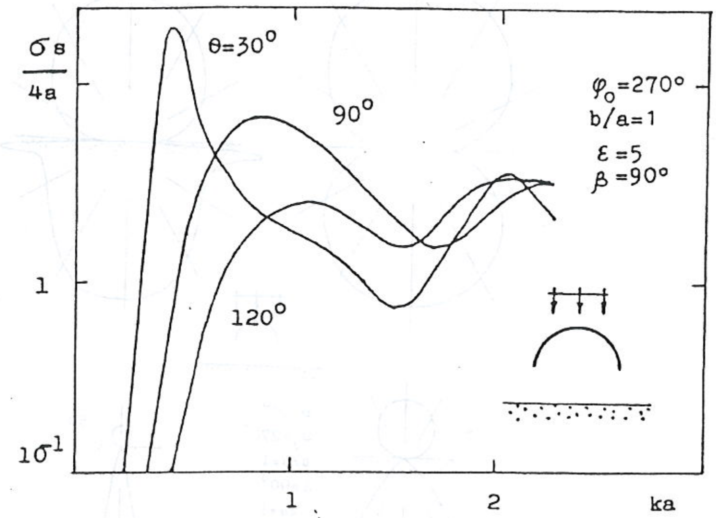

Fig. 9. Frequency dependence of normalized total scattering cross-sections for open screens of various widths over dielectric halfspace.

Discretizing this expression in terms of μ_n coefficients for a circular screen, we have

$$\Phi^{\pm}(\varphi) = \sum_{(n)} \mu_n (-i)^n J_n'(ka) \begin{Bmatrix} e^{in\varphi} + \tilde{r}(\varphi) e^{i2kb\sin\varphi - in\varphi}, & D_+ \\ \sin\varphi (1/\varepsilon - \cos^2\varphi)^{-1/2} \tilde{t}(\varphi) e^{in\varphi}, & D_- \end{Bmatrix} \qquad (120)$$

Integrating the squared absolute values of the functions $\Phi^{\pm}(\varphi)$ with respect to φ, one can calculate the fractures of the TSC associated with the radiation to upper and lower halfspaces.

In Fig.9 the plots of TSC of a circular open screen are shown as functions of the dimensionless frequency parameter ka. Similarly to the free-space scattering (Section 3.4), one can see the resonant behaviour of the screen provided that the latter forms a sort of cavity. The first low-frequency peak of TSC is again due to the Helmholtz mode of a cavity-backed aperture. The other peaks are associated with higher-order natural damped H-modes of the same slitted cavity. New feature is that the scattered power can be enhanced due to phase-matching between the field reradiated from the screen itself and from its image located at the spacing $2b$ down the inreface.

Figs.10 and 11 show the normalized far field scattering patterns $\Phi^{\pm}(\varphi)$ of the screens with different angular widths placed over the dielectrics of different contrasts. First of all, there is no scattering along the interface $\Phi^{\pm}(\varphi) \to 0$ as $\varphi \to 0, \pi$. Besides, there are two simmetrically directed sidelobes of the patterns clearly observable in the optically denser medium D_-. Their direction is conditioned by the angle of the total internal reflection (TIR)

$$\gamma_t = \arccos(\varepsilon^{-1/2}) \qquad (121)$$

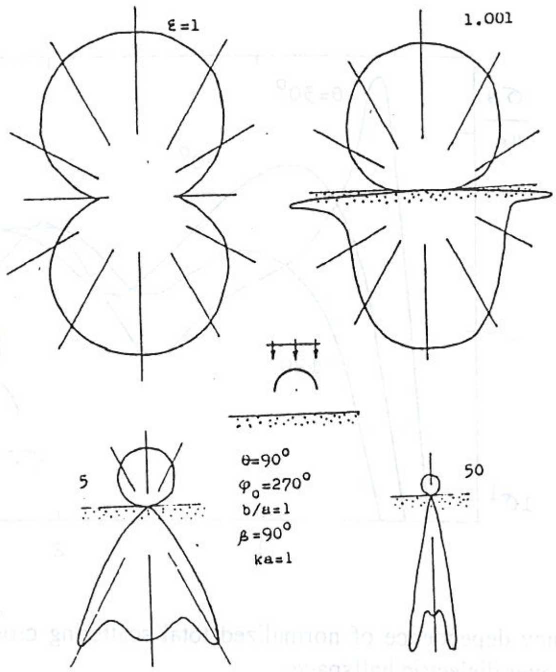

Fig. 10. Far-field scattering patterns for a semicircular screen over dielectric halfspace with various permittivities.

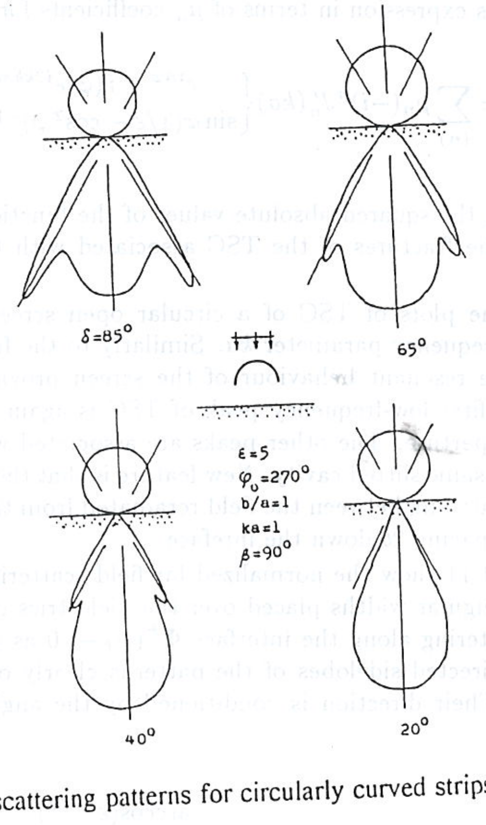

Fig. 11. Far-field scattering patterns for circularly curved strips of various widths over dielectric halfspace.

So, the higher the contrast, the closer TIR-lobes to the normal direction independently on the angle of incidence β, and hence, the narrower the whole pattern $\Phi^-(\varphi)$. These TIR-lobes are present for any scatterer either elevated over or buried in the dielectric halfspace. In general, they are more in amplitude for larger contrasts and larger dimensions of the scattering object.

4.3 Impedance-Plane Surface Wave Scattering from a Screen-Shaped Inhomogeneity

In this section we consider another example of H-wave scattering from a screen placed in a simple stratified medium. Unlike the previous section, this one deals with an impedance plane boundary which can support a guided undecaying natural mode of surface character. This fact leads to the modification of the radiation condition and results in appearing the transmission and reflection coefficients of the incident mode as quantities being sought.

The scattering geometry is presented in Fig.12. Here a screen M of the same parameters as before is located at the distance b over a plane boundary with impedance Z. Assume that the incident field is H-polarized, then the total field is of the same polarization due to the $2-D$ character of the problem, that is defining $H = H^{in} + H^{sc}$ as before, we have

$$(\Delta + k^2)H^{sc}(\vec{r}) = 0, \qquad \vec{r} \in D_+ \backslash M \tag{122}$$

$$\frac{\partial}{\partial n}(H^{in} + H^{sc}) = 0, \qquad \vec{r} \in M \tag{123}$$

$$\int_B (k^2|H^{sc}|^2 + |\nabla H^{sc}|^2) d\vec{r} < \infty, \qquad \forall B \in D_+ \tag{124}$$

plus the boundary condition that is to be satisfied at the plane $y = 0$

$$\left(\frac{\partial}{\partial y} + \alpha\right)(H^{in} + H^{sc}) = 0, \qquad \alpha = i\omega Z. \tag{125}$$

To make a correct statement we need first to analyze the eigenvalue problem for natural guided modes supported by the boundary. Seeking the nontrivial solutions of corresponding eigenvalue problem we obtain the dispersive equation

$$ig + \alpha = 0. \tag{126}$$

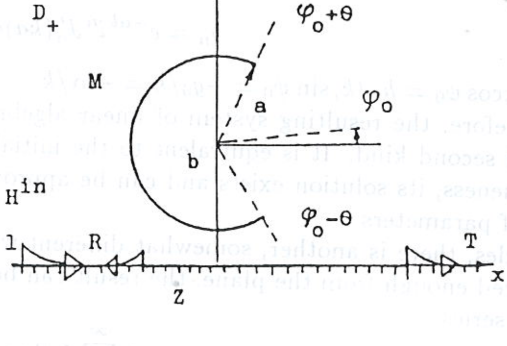

Fig. 12. Scattering geometry for a screen over impedance plane. Surface natural mode is incident.

Provided that α is real, the

Its single real positive solution is

$$h_0 = (k^2 + \alpha^2)^{1/2} \tag{127}$$

and yields the following surface mode guided by impedance plane

$$H_0(\vec{r}) = Ae^{-\alpha y + ih_0 x}. \tag{128}$$

Further we assume that it is this mode that is incident, so $H^{in} = H_0$ with $A = 1$. Then the needed condition at infinity takes the form

$$H^{sc}(\vec{r}) \underset{r\to\infty}{\sim} \Phi(\varphi)\left(\frac{2}{i\pi kr}\right)^{1/2} e^{ikr} + \begin{Bmatrix} T-1, & x>0 \\ R, & x<0 \end{Bmatrix} e^{-\alpha y + ih_0|x|} \tag{129}$$

where $0 < \varphi < \pi$, and the quantities T, R and $\Phi(\varphi)$ are to be found in our analysis.

To this end, using a double-layer-potential representation for the scattered field, one can derive an equation similar to (40) with the kernel defined by the Green's function of the halfplane D_+ bounded by impedance boundary

$$\begin{aligned} G(\vec{r}, \vec{r'}) &= G_0(\vec{r}, \vec{r'}) + G^{sc}(\vec{r}, \vec{r'}) \\ &= G_0(\vec{r}, \vec{r'}) + \frac{i}{4\pi}\int_C \frac{1}{g} r(h) e^{ig(y+y') + ih(x-x')} \end{aligned} \tag{130}$$

where the function

$$r(h) = \frac{ig - \alpha}{ig + \alpha} \tag{131}$$

is now for the impedance plane reflection coefficient of a generalized plane wave incident at a complex angle ψ: $\cos\psi = h/k, \sin\psi = -g/k$.

Then the derivation of the dual series equations and their regularization by means of the RHP technique retraces the steps discussed in the previous section. The resulting matrix equation is similar to (114). A_{mn} are again given exactly by (52), however for A^1_{mn} and C_m we obtain (115), (116) where now we should take

$$\Omega_{np}(kb, \alpha) = \frac{i^{n+p}}{\pi}\int_C \frac{1}{g} r(h) e^{i2gb + i(n+p)\psi} dh \tag{132}$$

and

$$b_n = e^{-\alpha b} i^n J'_n(ka) e^{in\psi_0} \tag{133}$$

where ψ_0:$\cos\psi_0 = h_0/k, \sin\psi_0 = -g_0/k = -i\alpha/k$.

As before, the resulting system of linear algebraic equations for $\{\mu_n\}_{n=-\infty}^{\infty}$ is of the Fredholm second kind. It is equivalent to the initial boundary-value problem and, due to the uniqueness, its solution exists and can be approximated through proper truncation for any set of parameters.

Besides, there is another, somewhat different way of numerical solution. If the screen is distanced enough from the plane, the result can be obtained by iterations, in the form of operator series

$$\mu = (I - A)^{-1}\sum_{n=0}^{\infty}\left[A^1(I - A)^{-1}\right]^n C \tag{134}$$

This procedure is equivalent to taking account of the successive reflections from the interface of the field scattered by a single screen. As the operator of the free-space scattering is invertible (see Section 3.2), $(I-A)^{-1}$ exists and is bounded. Besides, it does not depend on b while the functions $\Omega_{np}(kb,\alpha) = O(b^{-1/2})$ as $b \to \infty$, so it can be shown that the series (134) converges for large enough b.

To study the field far from the obstacle, we need to calculate the Fourier transform similar to (118). Unlike the previous section, here we have not only the branch-point singularities on the contour of integration at $h = \pm k$ but also poles of the integrand at $h = \pm h_0$. The far-field scattering pattern can be estimated at $kr\sin\varphi >> 1$ by the saddle-point technique similar to that employed in the previous section, and the result is

$$\Phi(\varphi) = \sum_{(n)} \mu_n (-i)^n J_n'(ka)\left[e^{in\varphi} + \tilde{r}(\varphi)e^{i2kb\sin\varphi - in\varphi}\right] \tag{135}$$

where

$$\tilde{r}(\varphi) = \frac{\alpha - ik\sin\varphi}{\alpha + ik\sin\varphi} \tag{136}$$

As can be easily seen, for $\varphi \to 0, \pi$ we have $\Phi(\varphi) \to 0$, thus along the boundary it is the surface wave $H_0(\vec{r})$ that solely carries the power.

To obtain the amplitudes of the guided mode at $x \to \pm\infty$ along the interface, one has to take account of the residues taken at the poles $h = \pm h_0$. Assume, e.g., that $x > 0$ and introduce new variable s such that $ih = -s + ik$. Then a new integral repsentation is obtained

$$H^{sc}(\vec{r}) = ie^{ikx}\int_{C'} F(y,s)e^{-sx}ds \tag{137}$$

with the integration path C' being the image of the real h-axis in the s-plane (see Fig.13). Taking the branch cuts from $s = 2ik$ to the left and from $s = 0$ to the right in parallel to the real axis, and deforming the contour by shifting it to the right, we can reduce (146) to the sum of the residue at $h = h_0$ and branch-cut integral along the loop C''. Thus, we have

$$H^{sc}(\vec{r}) \underset{x\to\infty}{\sim} (T-1)e^{-\alpha y + ih_0 x} + H_1^{bc}(\vec{r}) \tag{138}$$

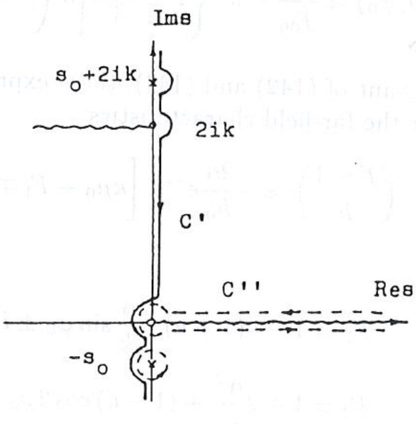

Fig. 13. Complex s-plane and contours of integration for asymptotic evaluation of the far field.

where

$$T - 1 = e^{-\alpha b}\frac{i\alpha}{h_0}\int_M \mu(\vec{r_c'})\frac{\partial}{\partial n_c'}e^{-ih_0x_c' - \alpha y_c'}d\vec{r_c'} \tag{139}$$

In the event that $x < 0$, we should shift the path C' to the left (Fig.13) and acting by analogy, we find

$$H^{sc}(\vec{r}) \underset{x\to-\infty}{\sim} = Re^{-\alpha y - ih_0x} + H_2^{bc}(\vec{r}) \tag{140}$$

where

$$R = e^{-\alpha b}\frac{i\alpha}{h_0}\int_M \mu(\vec{r_c'})\frac{\partial}{\partial n_c'}e^{ih_0x_c' - \alpha y_c'}d\vec{r_c'} \tag{141}$$

For a circular open screen, as we have, the discretization in terms of μ_n coefficients reduces the integration to summation with the result as

$$\begin{pmatrix} T-1 \\ R \end{pmatrix} = \frac{4\alpha}{h_0}e^{-\alpha b}\sum_{(n)}\mu_n(\mp i)^n J_n'(ka)\left(\frac{h_0}{k} \pm \frac{\alpha}{k}\right)^n \tag{142}$$

As for the additional terms $H_{1,2}^{bc}(\vec{r})$, we can expand the integrands in terms of asymptotic power series of s. First-term integration yields that $H_{1,2}^{bc} = O(|x|^{-1})e^{ik|x|}$ as $|x| \to \infty$. This fraction of the scattered field propagates with the free-space wavenumber k (unlike the surface-wave fraction) and is usually referred to as a lateral wave [46].

As it has been demonstrated in Section 3.5, linear algebraic equations obtained for the current density function coefficients are suitable for low-frequency asymptotic solutions. For the surface wave scattering, expanding all the quantities in terms of the power series of $\kappa = ka$, we arrive at asymptotic expressions uniform with respect to θ and φ_0

$$\mu_0 \approx -\frac{i\pi\kappa^2 e^{-\alpha b}}{D(\kappa,\theta,\varphi_0)}\left\{\kappa - \left(\frac{\alpha}{k}\sin\varphi_0 + i\frac{h_0}{k}\cos\varphi_0\right)\frac{1+u}{\tilde{T}_{00}} - \kappa\left[\left(1 + \frac{2\alpha^2}{k^2}\right)\cos 2\varphi_0 - 2i\frac{\alpha h_0}{k^2}\sin 2\varphi_0\right]\frac{1 - 2u - 3u^2}{8\tilde{T}_{00}}\right\} \tag{143}$$

$$\mu_{\pm1} \approx \mu_0\kappa^2\tilde{T}_{\pm10} - \frac{1}{2}\pi\kappa^2 e^{-\alpha b}\left(e^{i\psi_0}\tilde{T}_{\pm11} + e^{-i\psi_0}\tilde{T}_{\pm1-1}\right) \tag{144}$$

$$D(\kappa,\theta,\varphi_0) \approx \frac{1}{\tilde{T}_{00}} - \kappa^2\left\{1 + \frac{i\pi\kappa}{4}\left[\kappa\left(1 + \Omega_{00} + \Omega_{02}\frac{\tilde{T}_{02}}{\tilde{T}_{00}}\right) - 2\Omega_{01}\frac{\tilde{T}_{01}}{\tilde{T}_{00}}\right]\right\} \tag{145}$$

Taking account of (142) and (144), these expressions yield the following low-frequency asymptotics for the far-field characteristics

$$\begin{pmatrix} T-1 \\ R \end{pmatrix} \approx -\frac{2\alpha}{h_0}e^{-\alpha b}\left[\kappa\mu_0 - F_1 \mp \frac{i\pi\kappa^2}{2}(1+u)e^{-\alpha b}\begin{pmatrix} F_2 \\ F_3 \end{pmatrix}\right] \tag{146}$$

where

$$F_1 = 1 - \kappa(1+u)\left(\frac{\alpha}{k}\sin\varphi_0 \pm i\frac{h_0}{k}\cos\varphi_0\right)$$

$$F_2 = 1 + 2\frac{\alpha^2}{k^2} + (1-u)\cos 2\varphi_0$$

$$F_3 = 1 + (1-u)\left[\left(1 + \frac{2\alpha^2}{k^2}\right)\cos 2\varphi_0 + 2i\frac{\alpha h_0}{k^2}\sin 2\varphi_0\right]$$

and

$$\Phi(\varphi) \approx -\frac{1}{2}\kappa\mu_0 + \frac{i}{2}\kappa^2(1+u)\left[F_4\cos\varphi + F_5\sin\varphi + \tilde{r}(\varphi)e^{2kb\sin\varphi}(F_4\cos\varphi - F_5\sin\varphi)\right] \tag{147}$$

where

$$F_4 = \mu_0\cos\varphi_0 + \frac{\pi}{2}e^{-\alpha b}\left[\frac{h_0}{k} + (1-u)\left(\frac{h_0}{k}\cos 2\varphi_0 + i\frac{\alpha}{k}\sin 2\varphi_0\right)\right]$$

$$F_5 = \mu_0\sin\varphi_0 + \frac{\pi}{2}e^{-\alpha b}\left[i\frac{\alpha}{k} + (1-u)\left(\frac{h_0}{k}\sin 2\varphi_0 + i\frac{\alpha}{k}\cos 2\varphi_0\right)\right]$$

It is interesting to note that these formulas describe properly an electromagnetic analogue of so-called Dean's effect discovered initially in the dynamics of incompressible fluid [47, 37] : there is no reflection of a surface wave from a small submerged circular cylinder. Indeed, if we set $\theta = 0$ in the above formulas, we find that

$$\begin{pmatrix} T-1 \\ R \end{pmatrix} \approx \frac{i\pi\alpha}{h_0}e^{-2\alpha b}\begin{pmatrix} 4\alpha^2a^2 + \kappa^2 \\ -3\kappa^2 \end{pmatrix} \tag{148}$$

$$\Phi(\varphi) \approx -\frac{i\pi}{4}\kappa^2 e^{-\alpha b}\left\{\left[1 + \tilde{r}(\varphi)e^{i2kb\sin\varphi}\right]\left(1 - 2\frac{h_0}{k}\cos\varphi\right) - i\left[1 - \tilde{r}(\varphi)e^{i2kb\sin\varphi}\right]2\frac{\alpha}{k}\sin\varphi\right\} \tag{149}$$

and for $\kappa = ka \to 0$ while $\alpha a = const$ we obtain $R \to 0$ with nonzero $T - 1$.

In the event that $\theta \neq 0$ but κ is not large, the surface wave is scattered from a circular-cavity-backed slit aperture. Such a cavity, as it was shown in Section 3.5, has a special low-frequency natural damped mode of oscillation H_{00} referred to as the Helmholtz one. Analizing the eigenvalue problem for the matrix $||I - A(k) - A^1(k)||$ at $\kappa = ka < 1$, we come to the resonant frequency (73) perturbed by the presence of impedance plane

$$k_{00}^2 \approx a^{-2}(-2\ln\sin\frac{\theta}{2})^{-1}\left[1 + \frac{i}{8\pi}(1+\Omega_{00})\ln^{-1}\sin\frac{\theta}{2}\right] \tag{150}$$

So, if $\theta \to 0$ the Helmholtz mode frequency is located well within the low-frequency region. For this resonance we find from (146), (147) that both reflection and radiation are much more pronounced, namely

$$T - 1 \approx R \approx \frac{4\alpha_{00}}{h_0}e^{-2\alpha_{00}b}F_6 \tag{151}$$

$$\Phi(\varphi) = e^{-\alpha_{00}b}F_6\left[1 + \tilde{r}(\varphi)e^{i2k_{00}b\sin\varphi}\right] \tag{152}$$

where

$$F_6 = [1 - 2(\alpha_{00}a\sin\varphi_0 - ih_0a\cos\varphi_0)](1 + Re\Omega_0)^{-1} \tag{153}$$

Besides, an estimate similar to (81) justifies the solution by iterations for finite ka and $\delta = \pi - \theta \to 0$ i.e. for a narrow strip. Expanding functions $T_{mn}(-\cos\theta)$ in terms of asymptotic series of δ, we obtain

$$\begin{pmatrix} T-1 \\ R \end{pmatrix} \approx \pm\frac{4i\pi\alpha}{h_0}\sin^2\frac{\delta}{2}e^{-2\alpha(b-a\sin\varphi_0)}\begin{pmatrix} F_7 \\ F_8 \end{pmatrix} \tag{154}$$

where

$$F_7 = (\alpha a)^2 + \kappa^2 \cos^2 \varphi_0$$
$$F_8 = \left[(\alpha a)^2 \cos 2\varphi_0 + \kappa^2 \cos^2 \varphi_0 - 2i\alpha h_0 a \cos \varphi_0\right] e^{-2ih_0 a^2}$$

and

$$\Phi(\varphi) \approx -\pi\kappa \sin^2 \frac{\delta}{2}(\alpha a \sin \varphi_0 - ih_0 a \cos \varphi_0)e^{-ih_0 a \cos\varphi_0 - \alpha(b - a \sin \varphi_0)}$$
$$\left[\cos(\varphi - \varphi_0)e^{i\kappa \cos(\varphi-\varphi_0)} + \tilde{r}(\varphi)\cos(\varphi + \varphi_0)e^{i\kappa\cos(\varphi+\varphi_0)+2ikb\sin\varphi_0}\right] \quad (155)$$

These formulas agree with low-frequency asymptotics obtained above.

4.4 Mode Conversion and Scattering due to Screens in Dielectric-Slab Waveguide

Inhomogeneous dielectric-slab waveguides are known as basic elements of optical and microwave integrated circuits and antennas. Because of the open nature of the waveguide, the scattering of natural modes from an inhomogeneity is always accompanied not only by mode conversion but also by the radiation. The geometry of the problem analyzed in this section is illustrated in Fig. 14. The open waveguide is formed by a plate-parallel slab D_ε of thickness $2d$ and dielectric constant ε sandwiched between two free halfspaces $D_\pm = (y > d, y < -d)$. The slab is known to support the finite number of natural guided modes of two types TE_j and TM_j denoted from the viewpoint of the field structure with respect to the direction of propagation. Assume that TM_j mode is incident normally on the inhomogeneous section of the slab housing cylindrical screen-shaped obstacles with their axis at the same spacing b off the middle of the slab. Then it is obvious that here we have a case of H-polarized scattering with respect to the axis of cylinders. Thus, the incident field function can be found from the treatment of the corresponding eigenvalue problem that yields

$$H^{in}(\vec{r}) = V_j^{e,o}(y)e^{ih_j x} = \begin{Bmatrix} v_j^{e,o}(p_j y), & |y| < d \\ v_j^{e,o}(p_j d)e^{ig_j(|y|-d)}, & |y| \geq d \end{Bmatrix} e^{ih_j x} \quad (156)$$

where $g_j = (k^2 - h_j^2)^{1/2}, p_j = (k^2\varepsilon - h_j^2)^{1/2}$, $v^e(\cdot) = \cos(\cdot), v^o(\cdot) = \sin(\cdot)$ and the mode wavenumber $h_j : k < h_j < k\varepsilon^{1/2}$ satisfies one of the dispersion equations for even TM_{2n} or odd TM_{2n+1} modes of the slab $(n = 0, 1, 2, \ldots)$.

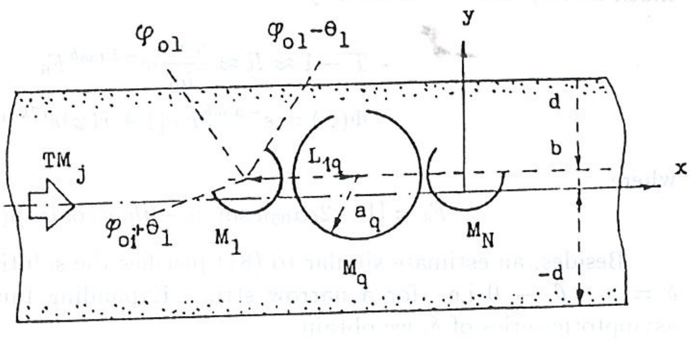

Fig. 14. Scattering geometry for a collection of screens inside a dielectric-slab waveguide. Natural guided mode is incident.

$$\Delta^e(h) \equiv ig\varepsilon \cos pd + p \sin pd = 0, \tag{157}$$

$$\Delta^o(h) \equiv ig\varepsilon \sin pd - p \cos pd = 0, \tag{158}$$

Decomposing the total field into the incident and the scattered field terms, we obtain for the latter a boundary value problem

$$\left[\Delta + k^2\varepsilon(\vec{r})\right] H^{sc}(\vec{r}) = 0, \qquad \vec{r} \in R^2 \backslash (M, y = \pm d) \tag{159}$$

where $\varepsilon(\vec{r}) = \varepsilon$ for D_ε and 1 elsewere, $M = \bigcup_{s=1}^{N} M_s$

$$[H^{sc}] = 0, \qquad \left[\frac{1}{\varepsilon}\frac{\partial H^{sc}}{\partial y}\right] = 0, \qquad y = \pm d \tag{160}$$

$$\frac{\partial}{\partial n_s}(H^{in} + H^{sc}) = 0 \qquad \vec{r} \in M_s, \qquad s = 1\ldots, N \tag{161}$$

$$\int\limits_B \left(k^2\varepsilon|H^{sc}|^2 + |\nabla H^{sc}|^2\right) d\vec{r} < \infty, \qquad \forall B \subset R^2 \tag{162}$$

$$H^{sc}(\vec{r}) \underset{r\to\infty}{\sim} \begin{Bmatrix} \Phi_j^{\pm}(\varphi)\left(\frac{2}{i\pi kr}\right)^{1/2} e^{ikr}, & y > d, y < -d \\ 0, & |y| < d \end{Bmatrix}$$

$$+ \sum_{q=0}^{Q} \begin{Bmatrix} T_{qj} - \delta_{qj}, & x > 0 \\ R_{qj}, & x < 0 \end{Bmatrix} V_q(y) e^{ih_j|x|} \tag{163}$$

Our aim is to find an effective analytical-numerical solution of this problem resulting in calculation of the mode conversion coefficients T_{qj}, R_{qj} and far-field scattering pattern $\Phi_j^{\pm}(\varphi)$ with a guaranteed accuracy. Seek the scattered field as a sum of N generalized double-layer potentials

$$H^{sc}(\vec{r}) = \sum_{s=1}^{N} H_s^{sc}(\vec{r}) = \sum_{s=1}^{N} \int\limits_{M_s} \mu^s(\vec{r}_s)\frac{\partial}{\partial n_s} G(\vec{r}, \vec{r}_s) d\vec{r}_s. \tag{164}$$

Here $\mu^s(\vec{r})$ is the induced current density function for the s-th screen, $G(\vec{r}, \vec{r}_s)$ is the Green's function of the slab which satisfies a problem similar to (159),(160),(163) but with a Dirac delta-function in the right-hand part of (159). As all the obstacles are inside the slab, we can set $|y_s| < d$ and derive that

$$G(\vec{r}, \vec{r}_s) = \frac{i}{4}\delta_{j\varepsilon} H_0^{(1)}\left(k\varepsilon^{1/2}|\vec{r} - \vec{r}_s|\right) + \frac{i}{4\pi}\int\limits_C F_j(y, y_s, h) e^{ih(x - x_s)} dh \tag{165}$$

where index j takes the values $+, -$ or ε depending on the location of the observation point $\vec{r} \in D_\pm$ or D_ε, respectively. The Fourier transforms are found to be

$$F_+ = e^{ig(y-d)}\left\{\left(1 - \frac{\varepsilon g}{p}\right) e^{ipd}\left[\frac{i\cos py_s \cos pd}{\Delta^e(h)} + \frac{\sin py_s \sin pd}{\Delta^o(h)}\right] + \frac{e^{ip(d-y_s)}}{p}\right\} \tag{166}$$

$$F_\varepsilon = \left(1 - \frac{\varepsilon g}{p}\right) e^{ipd} \left[\frac{i \cos py_s \cos py}{\Delta^e(h)} + \frac{\sin py_s \sin py}{\Delta^o(h)}\right] \tag{167}$$

$$F_- = e^{-ig(y+d)} \left\{\left(1 - \frac{\varepsilon g}{p}\right) e^{ipd} \left[\frac{i \cos py_s \cos pd}{\Delta^e(h)} - \frac{\sin py_s \sin pd}{\Delta^o(h)}\right] + \frac{e^{ip(y_s+d)}}{p}\right\} \tag{168}$$

with $g = (k^2 - h^2)^{1/2}$, $p = (k^2\varepsilon - h^2)^{1/2}$, $Img \geq 0$, $Reg \geq 0$. Note that the Fourier transforms have only two branch points at $h = \pm k$ and at least one pair of real poles $h = \pm h_0$ at the contour of integration C. These poles correspond to the principal (even) TM_0 mode having no low-frequency cutoff with the wavenumber given by the lowest zero of $\Delta^e(h)$. Next real poles at $h = \pm h_1$ correspond to the first higher-order (odd) mode TM_1 and appear as the lowest zero of $\Delta^0(h)$ provided that $kd(\varepsilon - 1)^{1/2} > \pi/2$, and so on.

To find the unknown functions $\mu^s(\vec{r})$ subject to the boundary condition (161), derive the set of N coupled integro-differential equations valid on curves M_s $(s = 1, \ldots N)$ and seek the solution satisfying (162). To reduce the problem to the dual series equations we introduce angular Fourier-series expansions in local coordinates of each of the cylinders, e.g.

$$\mu^s(\varphi_s) = \frac{2}{i\pi k\varepsilon^{1/2} a_s} \sum_{(n)} \mu_n^s e^{in\varphi_s}, \quad s = 1, \ldots N \tag{169}$$

with unknown coefficients μ_n^s $(n = 0, \pm 1, \ldots)$, and also introduce similar expansions with known coefficients for the Green's function (165) and the incident field (156). Tracing term-by-term operations of the previous sections we arrive at N systems of dual series equations

$$\sum_{(n)} \left\{ \mu_n^s \left[J'_{ns} H'_{ns} e^{in\varphi_s} + J'_{-ns} \sum_{(m)} J'_{ms} \Omega_{nm} e^{im\varphi_s} \right] + b_n^s e^{in\varphi_s} \right.$$
$$+ \sum_{q=1, \neq S}^{N} \mu_n^q \left[J'_{nq} \sum_{(m)} H^{(1)}_{n-m}(k\varepsilon^{1/2} L_{qs}) J'_{mq} e^{i(n-m)\varphi_{qs} + im\varphi_s} \right.$$
$$\left.\left. + J'_{-nq} \sum_{(t)} \Omega_{nt} \sum_{(m)} J_{t-m}(k\varepsilon^{1/2} L_{qs}) J'_{mq} e^{i(t-m)\varphi_{qs} + im\varphi_s} \right] \right\} = 0, \quad \varphi_s \in M_s \tag{170a}$$

$$\sum_{(n)} \mu_n^s e^{in\varphi_s} = 0, \quad \varphi_s \in S_s \tag{170b}$$

with $J'_{ns} = J'_n(k\varepsilon^{1/2} a_s)$, $H'_{ns} = H^{(1)'}_n(k\varepsilon^{1/2} a_s)$. L_{qs} is the distance between the centers of q-th and s-th screens, $\varphi_{qs} = 0$ if $q < s$ and π if $q > s$ and

$$\Omega_{nt}(kd, kb, \varepsilon) = \frac{i^{n+t}}{\pi} \int_C \left(1 - \frac{\varepsilon g}{p}\right)$$
$$e^{ipd} \left[\frac{i \cos(n\psi + pb) \cos(t\psi + pb)}{\Delta^e(h)} + \frac{\sin(n\psi + pb) \sin(t\psi + pb)}{\Delta^o(h)}\right] dh \tag{171}$$

$$b_n^s = i^n J'_{ns} v^{e,o}(n\psi_j + p_j b) e^{inh_j L_{s1}} \tag{172}$$

where the function $\psi(h)$ is defined as usually, through expressions $\cos\psi = h/k\varepsilon^{1/2}$, $\sin\psi = -p/k\varepsilon^{1/2}$, while $\psi_j = \psi(h_j)$.

Besides the dual series equations the coefficients must satisfy the set of conditions

$$\sum_{(n)} |\mu_n^s|^2 |n+1| < \infty, \qquad s = 1, \ldots, N \tag{173}$$

ensuring the field behaviour in agreement with the edge condition.

As we already know equations like (170) can be effectively regularized by means of the partial inversion procedure based on the RHP technique. As before, the term inverted analytically corresponds to small argument behaviour of the product $J'_{ns}H'_{ns}$ in (170a), i.e. to the main term of the static part of the dual series operator. Resulting algebraic equations for unknown coefficients $\mu^s = \{\mu_n^s\}_{n=-\infty}^{\infty}, s = 1, \ldots, N$ can be written as

$$(I - A_{ss} - A_{ss}^1)\mu^s - \sum_{q=1,\neq s}^{N} (A_{sq} + A_{sq}^1)\mu^q = C^s \tag{174}$$

where $s = 1, \ldots, N$, and operator A_{ss} is for the single-screen-free-space scattering case, i.e. given by (52) with $a = a_s$ and $k\varepsilon^{1/2}$ instead of k, while A_{ss}^1 is formally the same as (115) with $a, k, \Omega_{n+p}^{+}(kb, \varepsilon)$ replaced by $a_s, k\varepsilon^{1/2}$ and $\Omega_{np}(kd, kb, \varepsilon)$, respectively, and interaction operators are

$$A_{sq} = \|i\pi\varepsilon(ka)^2 J'_{nq} \sum_{(p)} H_{n-p}^{(1)}(k\varepsilon^{1/2} L_{qs}) J'_{pq} e^{i(n-p)\varphi_{qs}} \tilde{T}_{mp}(\theta_s, \varphi_{0s})\|_{m,n=-\infty}^{\infty} \tag{175}$$

$$A_{sq}^1 = \|i\pi\varepsilon(ka)^2 J'_{nq} \sum_{(t)} \Omega_{nt} \sum_{(p)} J_{t-p}(k\varepsilon^{1/2} L_{qs}) J'_{pq} e^{i(t-p)\varphi_{qs}} \tilde{T}_{mp}(\theta_s, \varphi_{0s})\|_{m,n=-\infty}^{\infty} \tag{176}$$

The free terms in (174) are expressed through (172) according to (116) with similar substitutions.

The operators A_{sq}, A_{sq}^1 can be shown to be compact in the functional space l_2^N provided that the screens do not intersect each other and the surfaces of the slab. This guarantees the convergence of approximate solutions obtained through truncation to the exact one with increasing the matrix order. Moreover, this solution does belong to a narrower class governed by (173).

As for the far-field parameters, carrying out the derivations by analogy to the previous section, we must take account of finite number of real zeros of the function $\Delta^e(h)\Delta^o(h)$ located between k and $k\varepsilon^{1/2}$. Using the asymptotic integration technique we finally arrive at

$$\begin{pmatrix} T_{mj} - \delta_{mj} \\ R_{mj} \end{pmatrix} = \frac{2(g_m\varepsilon - p_m)e^{ip_m d}}{p_m \Delta'^{e,o}(h_m)} \sum_{(n)} (\pm i)^n v^{e,o}(p_m b \pm n\psi_m) M_n(\pm h_m) \tag{177}$$

$$\Phi_j^{\pm}(\varphi) = \frac{k \sin\varphi e^{i\tilde{p}d}}{\tilde{p}} \sum_{(n)} (-i)^n \left[(\varepsilon\tilde{g} - \tilde{p}) \left(\frac{\cos\tilde{p}d \cos(n\tilde{\psi} + \tilde{p}b)}{i\Delta^e(k\cos\varphi)} \mp \frac{\sin\tilde{d}\sin(n\tilde{\psi} + \tilde{b})}{i\Delta^o(k\cos\varphi)} \right) + e^{\mp i\tilde{p}b \mp in\tilde{\psi}} \right] M_n(k\cos\varphi) \tag{178}$$

where $g = -k\sin\varphi, \tilde{p} = -k(\varepsilon - \cos^2\varphi)^{1/2}, 0 < \varphi < \pi$ and the coefficients

$$M_n(h) = \sum_{s=1}^{N} \mu_n^s J'_{ns} e^{ihL_{s1}} \tag{179}$$

serve for the current induced on some "effective" circular scatterer (made of N ones) as it is seen from infinitely remote points inside and outside the slab.

The equations obtained are equally valid for arbitrary collection of screens, however we present further some numerical results for the scattering from a periodic grating of identical cylinders, i.e. $L_{qs} = L|q-s|$ where L is period of grating and $a_s = a, \theta_s = \theta, \varphi_{0s} = \varphi_0$ for all $s = 1, \ldots, N$.

Fig.15 shows the plots of the TM_0 mode transmission and reflection coefficients as functions of the relative screen radius a/d for a single-mode slab. Inside the slab there is only one screen shaped as a rather narrow cavity-backed aperture. The sharp resonant peaks observable at the plots detect the excitation of the familiar Helmholtz-type damped resonance H_{00} of the cavity. If the slab can support two guided modes, the additional resonances appear, as Fig.16 demonstrates. Their origination is explained by the higher-order damped resonances of the cavity which are splitted to two symmetrical families $H^{\pm}_{mn}$ (see Section 3.4). Note that for a symmetrically positioned screen ($b = 0, \varphi_0 = 0$ or π) these families can be excited separately by even and odd incident modes, that is not available under the free-space scattering by a screen.

Fig.17 gives the example of the dependence of the coefficients of mode conversion on the slot position angle φ_0. Here, there is a single semicircular screen placed into a three-mode slab, and principal TM_0 mode is incident. One can easily see that the plot of the transmission coefficient $T_{00}(\varphi_0)$ is symmetrical with respect to the point $\varphi_0 = \pi/2$. In other words, the incident mode transmission is exactly the same for two mirror-opposite obstacles but this is not true for other coefficients.

To explain this behaviour one needs to investigate the reciprocity relationship for open waveguide scattering. To this end, assume a guided natural mode of index either p or $-q$ to be incident from $x = -\infty$ or $x = \infty$ with the wavenumbers $h_p > 0$ and $h_{-q} = -h_q < 0$, respectively, and denote the total fields by the sums

$$H^{(+)} = H_p + H^{(+)sc}, \qquad H^{(-)} = H_{-q} + H^{(-)sc} \tag{180}$$

Then, apply the Green's formula (36) but with weight $\epsilon^{-1}(y)$ to functions $H^{(+)}$ and $H^{(-)}$ within the bounded domain D_* indicated in Fig.6. As both functions satisfy the modified condition of radiation (87), the final result of integration over $D_* \to \infty$ contains only the field products of the incident mode and the mirror-mode terms, that is

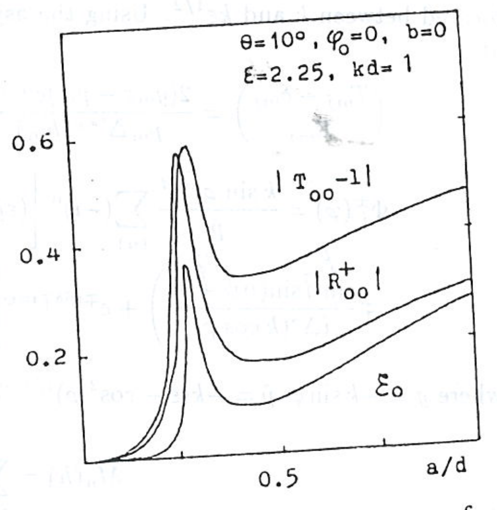

Fig. 15. Amplitudes of reflection and transmission coefficients as functions of relative radius of a cavity-shaped screen in a single-mode slab.

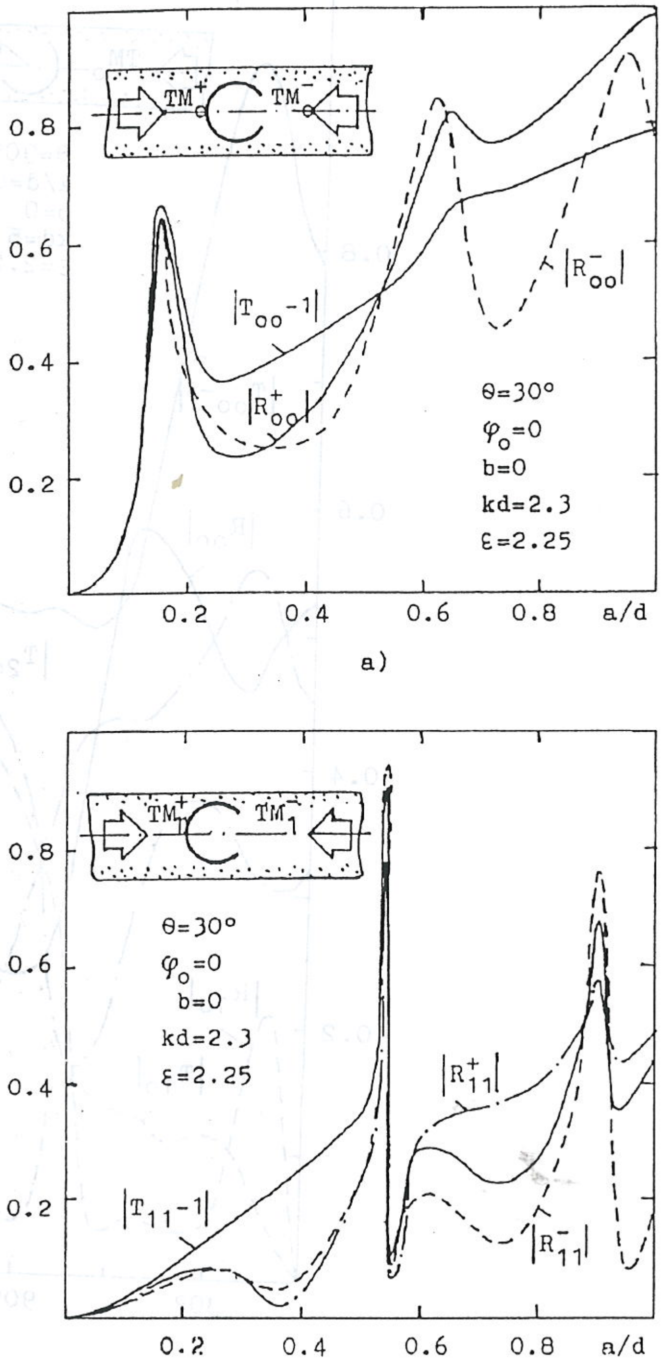

Fig. 16. Amplitudes of reflection and transmission coefficients versus relative radius of a cavity-shaped screen in a two-mode slab. First-even TM_0 (a) and first-odd TM_1 (b) modes are incident.

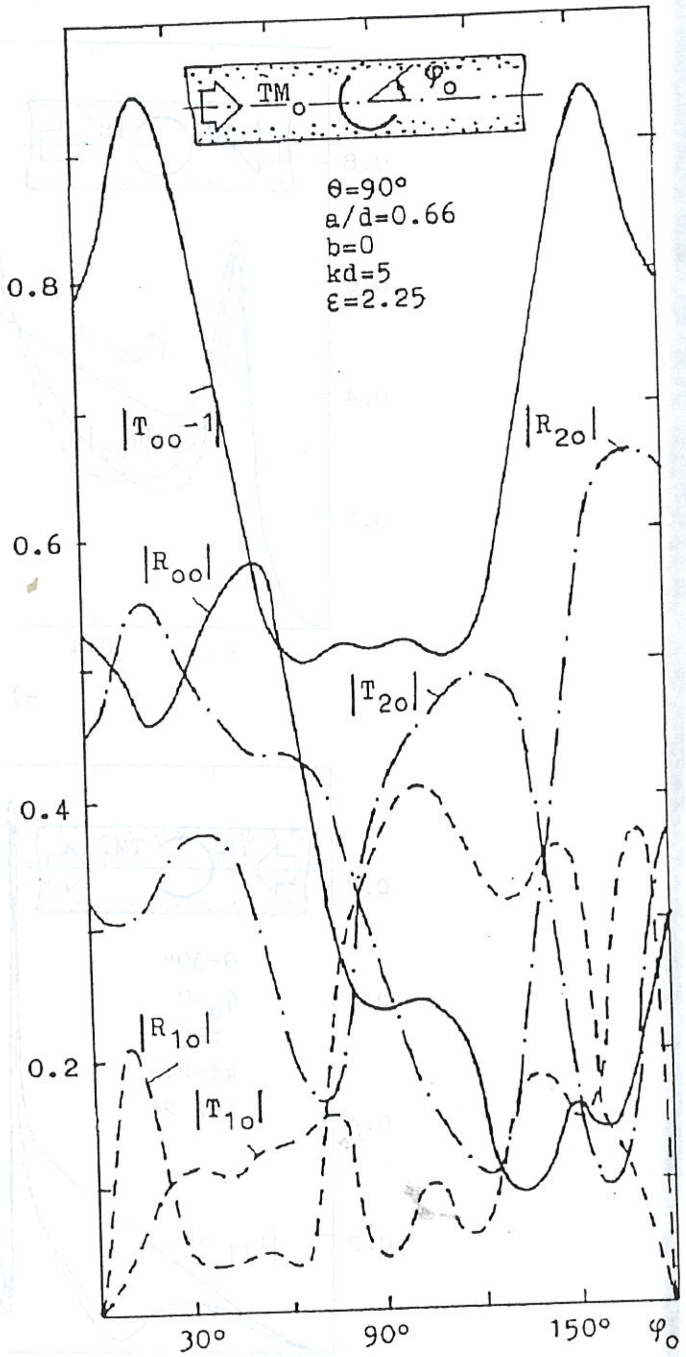

Fig. 17. Effect of screen's angular position angle ϕ_0 on the amplitudes of mode conversion coefficients. TM_0 mode of three-mode slab is incident.

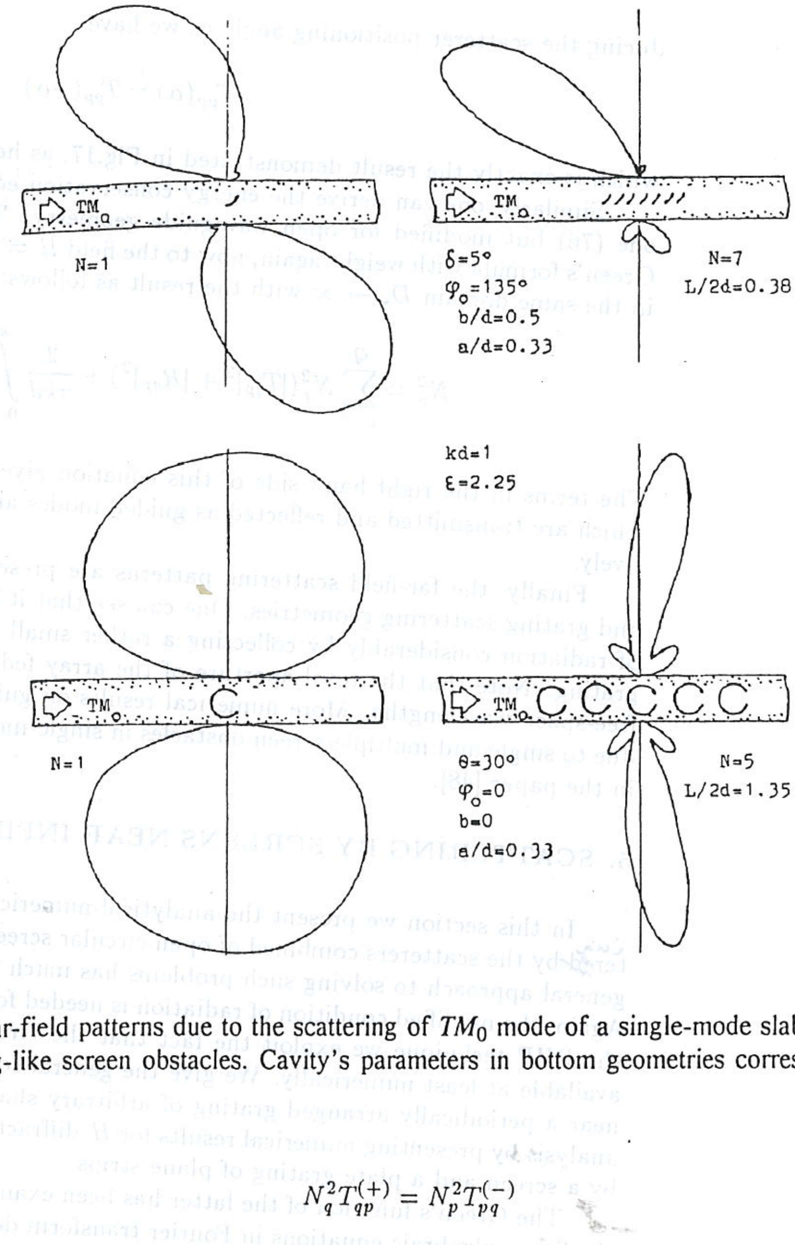

Fig. 18. Far-field patterns due to the scattering of TM_0 mode of a single-mode slab from single and grating-like screen obstacles. Cavity's parameters in bottom geometries correspond to H_{00} resonance.

$$N_q^2 T_{qp}^{(+)} = N_p^2 T_{pq}^{(-)} \tag{181}$$

where

$$N_j^2 = \frac{h_j}{kd} \int_{-\infty}^{\infty} \varepsilon^{-1} V_j^2 \, dy = \frac{h_j}{k} \left\{ \frac{1}{\varepsilon} + (-1)^j \frac{\sin 2p_j d}{2\varepsilon p_j d} + \frac{i[v^{e,0}(p_j d)]^2}{g_j d} \right\} \tag{182}$$

is the norm of the j-th guided mode.

Now it is obvious that for $q = p$ we obtain simply

$$T_{pp}^{(+)} = T_{pp}^{(-)} \tag{183}$$

Naturally, this fact is a counterpart of the well-known free-space identity of the forward-scattering amplitudes in the plane wave scattering (see Section 3.4 and (77)). Hence, introducing the scatterer positioning angle α, we have

$$T_{pp}(\alpha) = T_{pp}(-\alpha) \tag{184}$$

which is exactly the result demonstrated in Fig.17, as here $\alpha = \pi/2 - \varphi_0$.

Similarly, one can derive the energy conservation equation analogous to the free-space one (76) but modified for open waveguide geometry. To obtain it, one has to apply the Green's formula with weight again, now to the field $H = H_p + H^{sc}$ and its complex conjugate in the same domain $D_* \to \infty$ with the result as follows:

$$N_p^{2'} = \sum_{q=0}^{Q} N_q^2(|T_{qp}|^2 + |R_{qp}|^2) + \frac{2}{\pi kd}\int_0^{\pi}(|\Phi^+|^2 + |\Phi^-|^2)d\varphi \tag{185}$$

The terms in the right-hand side of this equation give the fractions of the incident power which are transmitted and reflected as guided modes and lost as a cylindrical wave, respectively.

Finally, the far-field scattering patterns are presented in Fig.18 for the single screen and grating scattering geometries. One can see that it is possible to enhance the directivity of radiation considerably by collecting a rather small number of scatterers into a periodic grating. Note that the total aperture of the array fed by the slab waveguide is about two free-space wavelengths. More numerical results on guided modes conversion and radiation due to single and multiple-screen obstacles in single-mode and multimode slab can be found in the paper [48].

5. SCATTERING BY SCREENS NEAR INFINITE PERIODIC GRATING

In this section we present the analytical-numerical method of solution for waves scattered by the scatterers combined of open circular screens and infinite periodic gratings. The general approach to solving such problems has much in common with the previous section. Again, the modified condition of radiation is needed for a correct statement, and when using the RHP technique we exploit the fact that the Green's functions of various gratings are available at least numerically. We give the general solution for a finite collection of screens near a periodically arranged grating of arbitrary shaped elements. Then we illustrate this analysis by presenting numerical results for H-diffraction by an open $2-D$ resonator formed by a screen and a plate grating of plane strips.

The Green's function of the latter has been examined by Skurlov [49] who reduced it to the linear algebraic equations in Fourier transform domain. The full combined problem was first treated by Nosich [50] who gave the formal solution. Some numerical results on plane and cylindrical wave scattering by screen over a short-periodic strip grating can be found in [43,51]. This analysis takes account of the grating approximately, within the framework of the Lamb's rule description of the Fourier transformed Green's function, but it can be deepened by using a full-wave numerical algorithm.

5.1 Formulation of the Problem and Derivation of Basic Equations

Consider the $2-D$ space R^2 containing an infinite periodic (of period L) diffraction grating located completely within a strip $D_g = \{-2p \leqslant y \leqslant 0\}$ referred to as a grating domain (see Fig.19). Let the upper and lower free-halfspaces $D_\pm$ contain $N_\pm$ perfectl

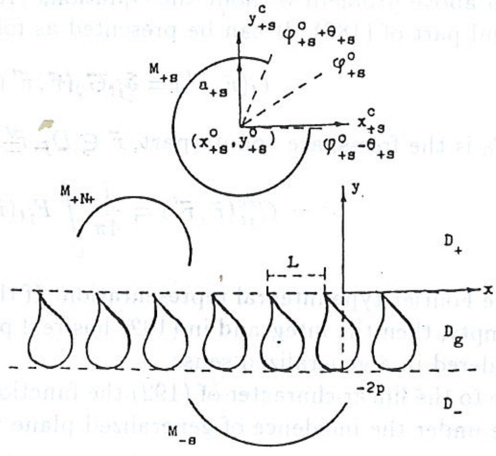

Fig. 19. On the scattering from a collection of screens near an infinite periodic grating.

conducting circular screens $M_{\pm s}$ $(s = 1, \ldots, N_j, \quad j = \pm)$ of the same geometry as in the previous sections characterized by the set of parameters $(a_{js}, \varphi_0^{js}, \theta_{js}, x_{js}^0, y_{js}^0)$.

The screens are assumed not to intersect each other or the boundaries of the grating domain. From the viewpoint of mathematics the problem of external excitation by some H-polarized field $H^{in} = H - H^{sc}$ is formulated as follows

$$(\Delta + k^2) H^{sc}(\vec{r}) = 0 \tag{186}$$

where $\vec{r} \in R^2 \backslash (M \cup D_g)$, $\quad M = \bigcup_{j=\pm, s=1}^{N} M_{js}, \quad N = N_+ + N_-, \quad Imk = 0,$

$$\tilde{D}_g H^{sc}(\vec{r}) = 0, \qquad \vec{r} \in D_g \tag{187}$$

$$\frac{\partial}{\partial n_s}(H^{in} + H^{sc}) = 0, \qquad \vec{r} \in M_{\pm s}, s = 1, \ldots, N_{\pm} \tag{188}$$

$$\int_B \left(k^2 |H^{sc}| + |\Delta H^{sc}|^2\right) d\vec{r} < \infty, \qquad \forall B \subset R^2 \backslash D_g \tag{189}$$

$$H^{sc}(\vec{r}) \underset{r \to \infty}{\sim} \left\{ \begin{array}{ll} \Phi^{\pm}(\varphi) \left(\frac{2}{i\pi k r}\right)^{\frac{1}{2}} e^{ikr}, & \vec{r} \in D_{\pm} \\ 0, & \vec{r} \in D_g \end{array} \right\} + \sum_{q=1}^{Q} \left\{ \begin{array}{ll} \alpha_q, & x > 0 \\ \beta_q, & x < 0 \end{array} \right\} V_q(y) e^{i\gamma_q h_q |x|} \tag{190}$$

Here the operator $\tilde{D}_g$ contains a set of boundary and edge conditions on the contours of grating elements. The condition at infinity (190) takes account of the grating's natural guided modes $V_g(y) e^{ih_q x}$ provided that they exist.

The general scheme of the treatment is similar to that given before. We start with seeking the scattered field as an assembly of the double-layer potentials like (164). The kernel function of such a representation is the normal derivative of the Green's function of the grating. So the initial problem is splitted into two ones: the search for the Green's function and the search for the currents on the screens. The Green's function satisfies the given above problem without the equations (188), (189) and with delta-function in the right-hand part of (186). It can be presented as follows

$$G(\vec{r},\vec{r}') = \delta_{jt}G_0(\vec{r},\vec{r}') + G^{sc}_{jt}(\vec{r},\vec{r}') \tag{191}$$

where G_0 is the free-space counterpart, $\vec{r} \in D_j, \vec{r}' \in D_t$, $j,t = \pm$ and

$$G^{sc}_{jt}(\vec{r},\vec{r}') = \frac{i}{4\pi}\int\limits_C F_{jt}(\vec{r},\vec{r}',h)e^{ihx}dh \tag{192}$$

being the Fourier-type integral representation. If the natural modes spectrum of the grating is not empty, then the integrand in (192) has real poles on C , and the whole integral should be considered in a generalized sense.

Due to the linear character of (192) the function $F_{jt}(\vec{r},\vec{r}',h)$ can be treated as a grating response under the incidence of generalized plane wave

$$\tilde{H}^0_{jt} = \begin{cases} \delta_{jt}A^0_j(\vec{r}',h)e^{-ig|y|+ihx}, & \vec{r} \in D_j \\ 0, & \vec{r} \in D_t \end{cases} \tag{193}$$

propagating from the halfspace D_j at the angle ψ: $\cos\psi = h/k$, $\sin\psi = -g/k$ with the amplitude

$$A^0_j(\vec{r}',h) = \frac{1}{g}e^{-ig|y'|-ihx'} \tag{194}$$

As a result, the function $F_{jt}(\vec{r},\vec{r}',h)$ satisfying the Helmholtz equation and conditions imposed in the grating domain D_g is obviously a periodic function of x with the period of L. Hence, off the domain D_g it can be expanded in terms of Floquet-Fourier series as follows

$$F_{jt}(\vec{r},\vec{r}',h) = A^0_j \sum_{(q)} \begin{Bmatrix} a^j_q(h), & \vec{r} \in D_j \\ b^j_q(h), & \vec{r} \in D_t \end{Bmatrix} e^{\frac{ikxq}{\kappa} + ig_q(|y-p|+p)} \tag{195}$$

where $\vec{r}' \in D_j, \kappa = L/\lambda, g_q = (k^2 - h_q^2)^{1/2}, h_q = h + qk/\kappa, Reg_q \geq 0, Img_q \geq 0$, and coefficients a^j_q, b^j_q, are the amplitudes of the field space harmonics produced by the plane wave like (193) but with a unit intensity.

The problem of seeking these amplitudes is an independent one, and the results are readily available with guaranteed accuracy for a wide amount of particular gratings (e.g., those of rectangular or circular bars, plane or inclined flat strips, nonclosed circular cylinders, etc.). As the expansion (195) gives the Fourier transform dependent on the coordinates x, y in explicit form, the further derivation can be made in a way similar to that of the Section 4.4. By expanding the surface current densities $\mu^{js}(\vec{r}_{js})$ and the integrand in the Green's function (192) in terms of angular exponents in the local coordinates of s-th cylinder in j-th halfspace, we arrive at the representation

$$\begin{aligned} H^{sc}_j(\vec{r}) = &\sum_{s=1}^{N_j}\sum_{(n)} \mu^{js}_n Z^{js}_n(r_{js}, a_{js})e^{in\varphi_{js}} \\ &+ \sum_{s=1}^{N_j}\sum_{(n)} \mu^{js}_n i^n J'_n(ka_{js})S^{js}_{(a)n}(\vec{r}_{js}) \\ &+ \sum_{s=1}^{N_t}\sum_{(n)} \mu^{ts}_n i^n J'_n(ka_{ts})S^{ts}_{(b)n}(\vec{r}_{ts}) \end{aligned} \tag{196}$$

where

$$Z_n^{js}(x,y) = \frac{i}{4}\begin{cases} J_n(kx)H_n^{(1)'}(ky), & x < y \\ H_n^{(1)}(kx)J_n'(ky), & x > y \end{cases} \tag{197}$$

$$S_{(a,b)n}^{js}(\vec{r}_{js}) = \frac{i}{4\pi}\int_C \frac{e^{in\psi}}{g}\sum_{(q)}\left\{a_q^j, b_q^j\right\} e^{ig_q(|y-p-y_{js}^0|+p)}e^{ih_q(x-x_{js}^0)}dh. \tag{198}$$

Now we have to transform this field to corresponding coordinates of a screen M_{js} and enforce the boundary condition (188). This yields a system of N coupled dual series equations similar to (170). Each particular dual series equation can be subjected to the RHP-based partial inversion procedure (see Section 2.3) resulting in a system of N coupled matrix operator equations

$$\left(I - A_{ss}^j - A_{ss}^{j1}\right)\mu^{js} - \sum_{l=1,\neq s}^{N_j}\left(A_{sl}^j + A_{sl}^{j1}\right)\mu^{jl} - \sum_{l=1}^{N_t} A_{sl}^{t1}\mu^{tl} = C^{js} \tag{199}$$

where $s = 1,\ldots,N_j; j,t = \pm, j \neq t$

$$A_{ss}^j = ||\Delta_m^{js}\tilde{T}_{mn}(\theta_{js},\varphi_0^{js})||_{m,n=-\infty}^{\infty}$$

$$A_{sl}^j = ||i\pi(ka_{js})^2 J_n'(ka_{js})\sum_{(p)} H_{p-n}^{(1)}(kR_{sl}^j)e^{i(n-p)\gamma_{sl}^j}\tilde{T}_{mp}(\theta_{js},\varphi_0^{js})||_{m,n=-\infty}^{\infty}$$

$$A_{sl}^{j1} = ||i\pi(ka_{js})^2 J_n'(ka_{jl})\sum_{(p)} J_p'(ka_{js})\Omega_{(a)pn}^{jsl}\tilde{T}_{mp}(\theta_{js},\varphi_0^{js})||_{m,n=-\infty}^{\infty}$$

$$A_{sl}^{t1} = ||i\pi(ka_{js})^2 J_n'(ka_{tl})\sum_{(p)} J_p'(ka_{js})\Omega_{(b)pn}^{tsl}\tilde{T}_{mp}(\theta_{js},\varphi_0^{js})||_{m,n=-\infty}^{\infty}$$

$$C^{js} = \left\{i\pi(ka_{js})^2 b_n(ka_{js})\tilde{T}_{mn}(\theta_{js},\varphi_0^{js})\right\}_{m=-\infty}^{\infty}$$

and

$$\Omega_{(a,b)mn}^{jsl} = \frac{i^{m-n}}{4\pi}\int_C \frac{e^{igb+in\psi}}{g}\sum_{(q)}\left\{a_q^j, b_q^j\right\}\cdot$$

$$\cdot\, exp\left\{ig_q\left[(y_{jl}^0 - y_{js}^0) \pm p + p\right] \pm im\psi_q + ih_q(x_{jl}^0 - x_{js}^0)\right\}dh. \tag{200}$$

Here we have denoted through $(R_{sl}^j,\ \gamma_{sl}^j)$ the coordinates of the M_{jl}-th screen center in the local coordinates of M_{js}-th screen.

As before, the coefficients μ_n^{js} are governed by the demand of convergence of the series analogous to (173). Investigation of the obtained matrix equations shows that the solution of exactly the needed type exists, is unique, and can be approximated by means of truncation. All this provided that the screens do not intersect each other or the grating domain. However, the latter restriction is not absolute and it needs a more careful treatment to be overcome.

5.2 Diffraction by a Screen near a Plane Strip Grating

The geometry of this particular scattering problem is illustrated in Fig.20. A screen M is located at the distance b over an infinite plane-periodic strip grating of period L, strip width D and shift parameter w. Here, the grating operator $\tilde{D}_g$ is reduced to

$$\frac{\partial}{\partial y}\left(H^{in} + H^{sc}\right) = 0 \tag{201}$$

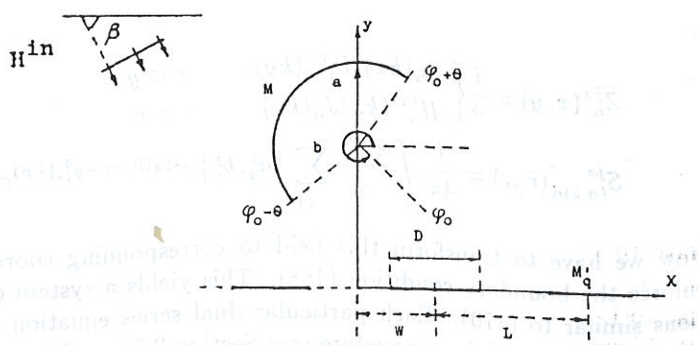

Fig. 20. Scattering geometry for a circular open screen over a periodic grating of flat strips. Plane wave is incident.

at $|x - w - nL| < D, y = 0$ for all $n = 0, \pm 1, \ldots$, and the condition like (189) requested in the vicinity of any strip edge.

Besides, such a zero-thickness perfectly conducting grating can be shown to support no guided natural modes along the x-axis, so the radiation condition (190) is reduced to Sommerfeld one.

As it has been demonstrated before, to obtain the basic equations we need first to find the Green's function of the grating $G(\vec{r}, \vec{r}')$. However, the latter needs a preceding treatment on the plane-wave scattering from the same grating. Here we have a screen above the grating, so we consider a plane H-wave incidence from D_+, that is

$$\tilde{H}^0 = \begin{cases} e^{ihx - igy_c}, & y_c = y - b > -b \\ 0, & y_c < -b. \end{cases} \qquad (202)$$

Besides, it is well known that for a plane strip grating there are valid the relations

$$b_0 = 1 - a_0, \qquad b_q = -a_q, \qquad (q = \pm 1, \pm 2, \ldots) \qquad (203)$$

which enable us to exclude the coefficients b_q from the further treatment.

The problem of obtaining the values of a_q is reducible to the dual series equations solvable by means of the RHP procedure (see [6–9,27,28]). To save the space we exploit the solution as it is given in [49].

If we introduce

$$\tilde{\theta} = \frac{\pi D}{L}, \qquad \tilde{\varphi}_0 = \frac{kw}{\kappa} \qquad (204)$$

and extract the integer number s from the quantity $\kappa h/k$ as $\kappa h/k = s + \gamma$, where $s = 0, \pm 1, \ldots, -1/2 < \gamma \leq 1/2$, then after denoting

$$m = s + q, \qquad a_m \equiv a_{m-s} = a_q \qquad (205)$$

we arrive at the matrix equations

$$(I - H)a = D \qquad (206)$$

where $a = \{a_m\}_{m=-\infty}^{\infty}, H = ||H_{mn}||_{m,n=-\infty}^{\infty}, D = \{D_m\}_{m=-\infty}^{\infty}$,

$$H_{mn} = r_n R_{mn}(\gamma) + \delta_{om} \gamma S_m(\gamma) \qquad (207)$$

$$D_m = -ir_0 R_{mq}(\gamma) \qquad (208)$$

and the following definitions are adopted

$$r_0 = i(g^2 - \gamma^2)^{1/2}, \qquad r_m = |m+\gamma| + i\sigma_m$$
$$\sigma_m \equiv \kappa g_m = k\left[g^2 - (m+\gamma)^2\right]^{1/2}$$
$$R_{mn} = e^{i(n-m)\tilde{\varphi}_0}\begin{cases} (m+\gamma)^{-1}V_{m-1}^{n-1}(\cos\tilde{\theta}), & m \neq 0 \\ W_\gamma^m(\cos\tilde{\theta},) & m = 0 \end{cases}$$
$$S_m = e^{-im\tilde{\varphi}_0}\begin{cases} (m+\gamma)^{-1}P_m(\cos\tilde{\theta}), & m \neq 0 \\ Q_\gamma(\cos\tilde{\theta}), & m = 0 \end{cases}$$
$$W_\gamma^m = \frac{\sin\pi\gamma}{\pi}\sum_{(p)}\frac{(-1)^p V_{p-1}^{m-1}(\cos\tilde{\theta})}{p+\gamma}$$
$$Q_\gamma = \frac{\sin\pi\gamma}{\pi}\sum_{(p)}\frac{(-1)^p P_p(\cos\tilde{\theta})}{p+\gamma}$$

and $P_n(\cdot)$ are for the Legendre polynomials.

The solution of (206) is shown to exist in l_2, is unique and may be approximated by solving a finite matrix equation of a large enough order. The coefficients behaviour $a_s(h) \sim O(|s|^{-3/2})$ as $|s| \to \infty$ guarantees the local energy limitation condition to be satisfied.

Let us return to the initial combined problem, assuming that the whole screen-grating structure is excited by the plane H-polarized wave incident at an angle β: $0 < \beta < \pi$, i.e.

$$H^{in}(\vec{r}) = \begin{cases} e^{ik(x_c\cos\beta - y_c\sin\beta)}, & y_c = y - b > -b \\ 0, & y_c < -b \end{cases} \tag{209}$$

Then according to the preceding analysis, the resulting matrix-operator equation takes the form as

$$(I - A - A^1)\mu = C \tag{210}$$

where the operator $A = \|A_{mn}\|_{m,n=-\infty}^{\infty}$ is again given by (52), while for the operator $A^1 = \|A_{mn}^1\|_{m,n=-\infty}^{\infty}$ the function of (111) in (115) should be replaced by

$$\Omega_{pn}^{(a)} = \frac{i^{p+n}}{\pi}\int_C \frac{e^{igb+in\psi}}{g}\sum_{(s)} a_s(h)e^{ig_sb+ip\psi_s}dh \tag{211}$$

As the incident field is that of a plane wave, the free term coefficients are

$$C_m = i\pi(ka)^2\sum_{(n)} J_n'(ka)(i^n + \Omega_n^0)\tilde{T}_{mn}(\theta,\varphi_0) \tag{212}$$

where

$$\Omega_n^0 = -i^n\sum_{(s)} a_s(k\cos\beta)e^{i(g_s - k\sin\beta)b + in\psi_s(\beta)} \tag{213}$$

When calculating the functions $\Omega_{pn}^{(a)}$ one can note that the leading contribution in the sum is from the terms for which $|h/k + s/\kappa| < 1$ as the rest terms give an exponentially small contribution at sufficiently large b. Besides, when integrating over h, the main contribution is from the interval $|h| < k$, and no residues from the poles appear. For these values of h

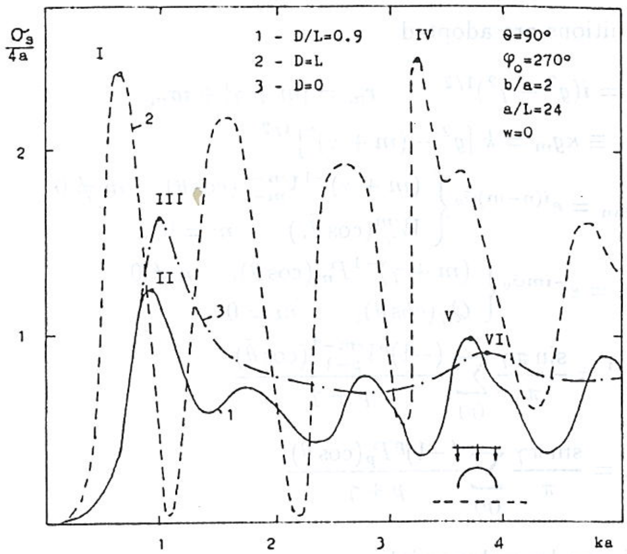

Fig. 21. Frequency dependence of normalized total scattering cross-sections of semicircular screen above strip grating, perfect reflector, and in free space.

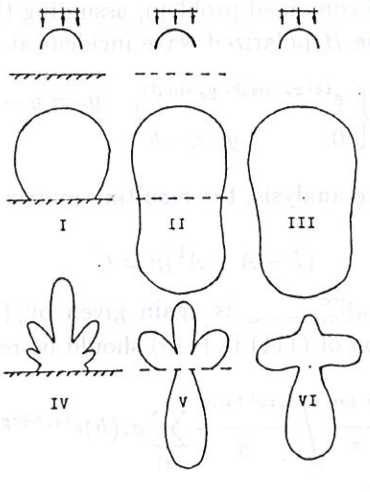

Fig. 22. Far-field scattering patterns calculated at the points marked in Fig. 21.

the energy conservation equation is valid. Hence

$$\sum_{s:|h/k+s/\kappa|<1} g^{-1}g_s(|a_s(h)|^2+|b_s(h)|^2) = 1-|a_0(h)|^2-|b_0(h)|^2 \qquad (214)$$

For a plane strip grating this is reduced to

$$2\sum_{s:|h/k+s/\kappa|<1} g^{-1}g_s|a_s(h)|^2 = 1-|a_0(h)|^2-|1-a_0(h)|^2 \qquad (215)$$

Therefore, if the grating parameters are chosen to provide a good penetration through the interface, i.e. $\sup|a_0(h)| \leq \Delta << 1$, then for the harmonics involved into (215)

$$|a_s(h)| \leq 1 - \Delta^2 - (1-\Delta)^2 = O(\Delta) \tag{216}$$

It follows from this fact that the whole operator A^1 is small compared to A, in the corresponding norm, and the interaction may be neglected.

In the opposite situation assume that the grating operates as a good reflector, and $\sup(1 - |a_0(h)|) \leq \Delta << 1$. Then extracting the 0-th term from the sum in (211), we conclude that $\Omega_{pn}^{(a)} = -H_{p-n}^{(1)}(2kb) + O(\Delta)$. It means that the resonator's behaviour is similar to that of a two-mirror one where the role of the second screen is played by the image of the first (real) screen M, so the separation $2b$ appears between the centres of curvature. Generally, a computation of the functions $\Omega_{pn}^{(a)}$ is rather a time-consuming procedure even for a simple grating. However, if the period is small compared to wavelength, i.e. $\kappa < 1$, a simple Lamb's rule is valid for any value of the stripwidth $D : 0 \leq D/L \leq 1$, that is

$$a_0(h) = \frac{i\kappa g Q}{1 - i\kappa g Q}, \qquad a_s(h) = 0(\kappa^2) \tag{217}$$

where

$$Q = -2 \ln \cos \frac{\pi D}{2L} \tag{218}$$

This approximation reduced the calculations to numerical integration of elementary functions, and it was used in further investigations. The values of interest were the far-field scattering pattern in the presence of grating and the total scattering cross-section. After implementation of the saddle-point asymptotic evaluation technique we find that

$$\Phi^{\pm}(\varphi) = \sum_{(n)} \mu_n (-i)^n J_n'(ka) \left[e^{in\varphi} \pm a_0(\pm k \cos\varphi) e^{-in\varphi} \begin{Bmatrix} e^{2ikb\sin\varphi} \\ 1 \end{Bmatrix} \right] \tag{219}$$

$$\sigma_s = \frac{2}{\pi k} \int_0^{\pi} \left(|\Phi^+|^2 + |\Phi^-|^2 \right) d\varphi \tag{220}$$

Figs.21 and 22 demonstrate the effect of a short-periodic strip grating on the scattering by a semicircular screen operating as a two-mirror open resonator (solid lines). For comparison, the dashed and dash-dotted curves correspond to the analogous characteristics of the same screen in free space and over a perfectly conducting plane, respectively. Regular resonances are observable at the frequencies producing phase-matching of the fields radiated from the screen and its image (two terms in (219)). Besides, some of the resonances are enhanced additionally provided that a natural oscillation mode of the screen itself is excited. One can see that by increasing the reflectivity of the interface the resonances are shifted to lower frequencies and become more intensive. As in the whole frequency range of the plots in Fig.21 the quantity $\kappa < 0.1$, the strip grating here is rather a poor reflector for an H-polarized wave. Therefore the far-field scattering patterns resemble those of the free-space diffraction (see Fig.22).

Similar approximate analysis based on the formulas analogous to (217) can be carried out for other shortperiodic gratings. Paper [52] contains the results on H-scattering from a screen in presence of a grating of rectangular grooves or of slitted circular cylinders. The latter case is remarkable for the grating operation as a highly frequency-selective surface, with a total reflection at the frequency of the Helmholtz mode of the elements but with an almost total transmission off this frequency.

6. CONCLUSION

In this paper the combined approach based on the Green's function and the Riemann-Hilbert Problem techniques has been discussed. It is used here for analyzing $2-D$ wave scattering from combined structures such as circular screens in presence of plane regular or regularly-periodic boundaries. If the plane wave diffraction from and penetration through these boundaries can be solved rigorously (as for dielectric slabs) or treated in a numerically accurate manner (as for various diffraction gratings), then the corresponding Green's functions are available in Fourier transform domain analytically or numerically. This fact gives the opportunity to exploit a modified procedure of the RHP-based partial inversion in wave scattering by a circular screen in such an inhomogeneous environment.

The discussed approach is remarkable for the full mathematical grounding, that is the solution is proved to exist, be unique and can be delivered numerically but with any desired degree of approximation.

Implementations of the approach are considered here only for H-polarized scattering. Of course, E-polarized case is of much interest as well. Besides, the screen-grating scattering analysis remains to be deepened by using the full-wave numerical algorithms taking account of grating properly. This can clarify many questions of practical interest, e.g. a behaviour of nonspecularly reflecting gratings placed in open resonators.

Moreover, the ideas discussed in this paper are applicable to other geometries of combined nature, e.g. like a curved screen and a semiplane or finite flat strip. One of the major shortcomings of the RHP method is known as inability to solve impedance-screen scattering problems. However, the impedance boundary conditions can be introduced on the flat strip and corresponding Green's function can be constructed by the modified Wiener-Hopf technique [53,54] or the orthogonal-polynomials method [55].

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